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Research Article

Characterization of Spectra of Posinormal Operators

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Abstract

Let H be a complex Hilbert space equipped with the inner product and let B(H) be the algebra of bounded linear operators acting on H. In this paper we have investigated the spectrum of an operator acting on a complex Hilbert space. In particular, we characterized the spectrum of a posinormal operator on an infinite dimensional complex Hilbert space. We also considered the point spectrum, the approximate point spectrum of a posinormal operator A and doubly commuting n-tuples of posinormal operators acting on a complex Hilbert space H. We have shown that Xia's property holds for a posinormal operator A. Finally, we have proved that doubly commuting n-tuples of posinormal operators are jointly normaloid.

Keywords: Spectrum; Posinormal operator; Hilbert Space; Spectral radius.

Introduction

Let H be a complex Hilbert space equipped with the inner product $\langle .,. \rangle$; and let $B(H)$ be the algebra of bounded linear operators acting on$H$. The Spectrum of a bounded linear operator A on a complex Hilbert space H is the set $\sigma(A) = \{\lambda: A - \lambda I \text{ is not invertible}\}$ [1-3]. The spectral radius of $A \in B(H)$ (denoted by $\gamma(A)$) is given by $\gamma(A) = sup\{|\lambda|: \lambda \in \sigma(A)\}[4,5]$. Spectral theory of linear operators on a Hilbert space was founded by Hilbert. In [6] they introduced the concept of a posinormal operator and gave a definition of it. He characterized an operator$A \in B(H)$, which is both positive $(\langle Ax, x\rangle \geq 0)$ and normal$(AA^* = A^*A)$. If $A \in B(H)$, is to be normal and positive, there must exist an interrupter $P \in B(H)$, such that$(AA^* = A^*PA)$, moreover A must be self adjoint. This result defines posinormality and their numerical solutions in cases where they ar operator equations [7].

In the present study we shall concentrate in characterizing the spectrum of a posinormal operator particularly the point spectrum and the approximate point spectrum of posinormal operators and commuting n-tuples of posinormal operators acting on a complex Hilbert space H.

The general problem being considered is, given a posinormal operator $A \in B(H)$ what can be said about its spectrum $\sigma(A)$?.

Preliminaries

In this section we give some of the basic properties of the spectrum of operators which are to be used in later discussions.

Theorem 2.1: ([10], Theorem 3.2)**.** Let H be a Hilbert space. Then there exists a Hilbert space $\mathfrak{K} \supset H$ and a map $\tau: B(H) \to B(\mathfrak{K})$ such that

(1) τ is an isometric algebraic *-isomorphism preserving the order; i.e., $\tau(A^*) = \tau(A^*)$,$\tau(\alpha A + \beta B) = \alpha\tau(A) + \beta\tau(B)$, $\tau(AB) = \tau(A)\tau(B)$, $\|\tau(A)\| = \|A\|$ and $\tau(A) \leq \tau(B)$ whenever $A \leq B$, for all $A, B \in B(H)$and $\alpha, \beta \in \mathbb{C}$;

(2) $\sigma(\tau(A)) = \sigma(A)$ and $\sigma_\pi(A) = \sigma_\pi(\tau(a))$ for all $A \in B(H)$, where $\sigma(A)$, $\sigma_p(A)$ and $\sigma_\pi(A)$ are the spectrum, the point spectrum and the approximate point spectrum of A, respectively.

Theorem 2.2: ([1], Theorem 2.1). For $A \in B(H)$ the following statements are equivalent:

(1) A is posinormal
(2) $Ran\ A \subseteqq Ran\ A^*$
(3) $AA^* \leq \lambda^2 A^*A$ for some $\lambda \geq 0$; and

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(4) There exists a $B \in B(H)$ such that $A = A^*B$. Moreover if (1), (2), (3) and (4) hold, there is a unique operator B such that:
(a) $\|B\|^2 = inf\{\mu: AA^* \leq \mu A^*A\}$;
(b) $KerA = KerB$ and
(c) $Ran\ B \subseteq (Ran\ A)^-$

Theorem 2.3: ([1], Theorem 3.1). Every invertible operator is posinormal.

Theorem 2.4: (6, Theorem 2). An operator T is posinormal if, and only if, there exists $\lambda > 0$ such that $|A(A|x, y)| \leq \lambda \| |A|x \|\| |A|y\|$ for all $x, y \in H$.

Theorem 2.5: ([11], Theorem 1.5). Let $A \in U|A|$ be a semi-hyponormal operator on H. If $Ax = r.e^{i\theta}x$ for a non-zero vector $x \in H$, then; $|A|x = rx$, $Ux = e^{i\theta}x$ and $A^*x = r.e^{-i\theta}x$.

Lemma 2.6: ([15], Theorem 4.2). Let $A = UP \in B(H)$, U be unitary, $P \geq 0$ and $A^*A = P^2$. Let $r > 0$, $|e^{i\theta}| = 1$. Then $r.e^{i\theta} \in \sigma_{n\pi}(A)$ if and only if there exists a sequence $\{x_k\}$ of unit vectors in H such that $\lim_{k\to\infty}\|(P - r)x_k\| = 0$ and $\lim_{k\to\infty}\|(U - e^{i\theta})x_k\| = 0$.

Theorem 2.7: ([12], Theorem 1.2). Let $\mathbb{A} = (A_1, \ldots, A_n)$ be an n-tuple of operators on H. Let τ be the mapping $\tau: B(H) \to B(\mathfrak{K})$ such that $\mathfrak{K}$ and H are Hilbert spaces and $\mathfrak{K} \supset H$. Then

$$\sigma_\pi(\mathbb{A}) = \sigma_\pi(\tau(\mathbb{A})) = \sigma_p(\tau(\mathbb{A})),$$

where $\tau(\mathbb{A}) = (\tau(A_1), \ldots, \tau(A_n))$.

Methodology

The methodology involved the use of known inequalities and techniques like the Cauchy-Schwarz inequality and the polarization identity [5]. To characterize the spectrum of posinormal operators we employed the technical approach of tensor products [12].

Results and discussion

Lemma 4.1: Let A be a posinormal operator. If $z \in \sigma_p(A)$ for $0 < p < \frac{1}{2}$ then $\bar{z} = \sigma_p(A^*)$

Proof. Suppose $0 \in \sigma_p(A)$.
Then there exists a non-zero vector $x \in H$ such that $Ax = 0$.
Since $|A|^2x = A^*Ax = 0$ and $|A| \geq 0$,
we have $(A^*A)^{\frac{1}{2}k}x = 0$ $(k = 1,2,\ldots)$.
For $m \in \mathbb{N}$ such that $\frac{1}{m} < p$,
we have $(A^*A)^{\frac{1}{2}m}x = 0$.

It follows that $(A^*A)^{\frac{1}{2}p}x = 0$. Clearly $(AA^*)^px = 0$ since A is posinormal. Therefore $A^*x = 0$. Next assume that $z \in \sigma_p(A)$ for non-zero $z \in \mathbb{C}$. Then there exists a non-zero vector $y \in H$ such that $Ay = zy$. Let $A = U|A|$be a polar decomposition of A with unitary operator U. Since $U|A|y = zy$, it follows that $|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}|A|^{\frac{1}{2}}y = z|A|^{\frac{1}{2}}y$.

We know that $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$.
Hence we have
$\tilde{A}^* = |A|^{\frac{1}{2}}U^*|A|^{\frac{1}{2}}y = \bar{z}.|A|^{\frac{1}{2}}y$.
Thus $A^*(|A|y) = \bar{z}.|A|y$. Since $|A|y \neq 0$, we have $\bar{z} = \sigma_p(A^*)$

Theorem 4.2: Let $A \in B(H)$ be a posinormal operator. Then $\sigma(A) = \{z: \bar{z} \in \sigma_\pi(A^*)\}$.

Proof.
Since we have $\sigma(A) = \sigma_\pi(A) \cup \{z: \bar{z} \in \sigma_\pi(A^*)\}$.
It suffices to show that $\sigma(A) = \{z: \bar{z} \in \sigma_\pi(A^*)\}$.
Assume that $z \in \sigma_\pi(A)$. Then we have $z \in \pi_p(\tau(A))$ where τ is a mapping. Suppose that $\tau(A)$ is posinormal, we have $\bar{z} \in \sigma_p(\tau(A^*))$. Also since $\bar{z} \in \sigma_p(\tau(A^*)) = \sigma_\pi(A^*)$, it follows that $\bar{z} \in \sigma_\pi(A^*)$.

Lemma 4.3: Let $\mathbb{A} = (A_1, \ldots, A_n)$ be doubly commuting n-tuple of posinormal operators on H. If $z = (z_1, \ldots, z_n) \in \sigma_p(\mathbb{A})$,
then $\bar{z} = (\overline{z_1}, \ldots, \overline{z_n}) \in \sigma_p(\mathbb{A}^*)$,
where $A^* = A_1^*, \ldots, A_n^*$.

Proof. There exists a non-zero vector $x \in H$ such that $A_ix = z_ix$ $(i = 1, \ldots, n)$. We assume that $z_1, \ldots, z_k$ are non-zero and $z_{k+1} = \cdots = z_n = 0$.
Therefore we obtain
$A_{k+1}^* = \cdots = A^*x = 0$.
Also $A_i^*(A_i|x) = \overline{z_i}.|A_i|x$, where A_{A_i} is the positive operator in a polar decomposition $A_i = U_i|A_i|$ $(i = 1, \ldots, k)$.
Suppose $|A_i| \ldots |A_k|x = 0$. Since then $(A_1 \ldots A_k)$ is doubly commuting k-tuple of a posinormal operator, U_i and $|A_i|$ commute with U_j and $|A_j|$ for every $i \neq j$. Thus we have $A_1.A_2 \ldots A_kx = 0$.
It follows that $z_1, \ldots, z_k = 0$.
Since every $z_i \neq 0$ $(i = 1, \ldots, k)$.
Therefore we have
$|A_i^*|(|A_1| \ldots |A_k|x) = |A_1| \ldots |A_{i-1}|.|A_{i+1}| \ldots |A_k|.A_1^*|A_i|x = \bar{z}_i(|A_i| \ldots |A_k|x)$.
Since also A_i commutes with $|A_1| \ldots |A_k|$, we have $|A_i^*|(|A_1| \ldots |A_k|x) = 0$ $(i = k+1, \ldots, n)$
Therefore it follows that $\bar{z}_i(\overline{z_1}, \ldots, \overline{z_n}) \in \sigma_p(\mathbb{A}^*)$.

Theorem 4.4: Let $\mathbb{A} = (A_1, \dots, A_n)$ be doubly commuting n-tuple of posinormal operators on H. Then $\sigma(A) = \{(z_1, \dots, z_n) \in \sigma_\pi(A^*)\}$.

Proof. Since $\mathbb{A}$ is doubly commuting n-tuple it follows that $(z_1, \dots, z_n) \in \sigma(A)$, there exists some partition $\{i_1, \dots, i_m\} \cup \{j_1, \dots, j_s\} = \{1, \dots, n\}$ and a sequence x_k of unit vectors in H such that $\{A_{i_\mu} - z_{i_\mu}\}x_k \to 0$ and $\{A_{j_v} - z_{j_v}\}^* x_k \to 0$ as $k \to \infty$, for $\mu = 1, \dots, m$ and $v = 1, \dots, s$.

Consider the mapping τ such that $\left(z_{i_1}, \dots, z_{i_m}, \overline{z_{j_1}}, \dots, \overline{z_{j_s}}\right) \in \sigma_\pi(\tau(B))$ where $\tau\,(B) = \left(\tau\left(A_{i_1}\right), \dots, \tau\left(A_{i_m}\right), \tau\left(A^*_{j_s}\right)\right)$.

Since $\tau(A_i)$ is a posinormal operator for every i $(i = 1, \dots, n)$ we have $(\overline{z_1}, \dots, \overline{z_n}) \in \sigma_p\left(\tau(A^*)\right)$. Therefore it follows that $(\overline{z_1}, \dots, \overline{z_n}) \in \sigma_\pi(A^*)$. Clearly $\sigma_\pi(A^*) \subset \sigma(A)$ and so the proof is complete.

Theorem 4.5: Let $\mathbb{A} = (A_1, \dots, A_n)$ be a doubly commuting n-tuple of posinormal operators on H. If $(r_1, \dots, r_n) \in \sigma(\mathbb{A}^*\mathbb{A}) \cup \sigma(\mathbb{A}\mathbb{A}^*)$, then there exists $(z_1, \dots, z_n \in \sigma(A)$ such that $|z_i|^2 \geq r_i \quad (i = 1, \dots, n)$,

where $\mathbb{A}^*\mathbb{A} = (A_1^* A_1, \dots, A_n^* A_n)$

and $\mathbb{A}\mathbb{A}^* = (A_1 A_1^*, \dots, A_n A_n^*)$.

Proof. We shall prove the theorem by induction. The theorem holds when $n = 1$.

We assume that the theorem holds for all doubly commuting $(n-1)$- tuple of posinormal operators. Assume that $(r_1, \dots, r_n) \in \sigma(\mathbb{A}^*\mathbb{A})$. Now since $\sigma(\mathbb{A}^*\mathbb{A}) = \sigma_\pi(\mathbb{A}^*\mathbb{A})$, we have by [13] and [14] that $(\sqrt{r_1}, \dots, \sqrt{r_n}) \in \sigma_\pi(|A|)$,

where $|\mathbb{A}| = (|A_1|, \dots, |A_n|)$.

Consider the mapping T : B(H) →B(H) such that $\sigma(T(A)) = \sigma(A)$ and $\sigma_\pi(A) = \sigma_\pi(T(A)) = \sigma_p(T(A))$ where $\sigma_\pi(A)$and $\sigma_p(A)$ are the approximate point spectrum and the point spectrum of A, respectively.

Let $\mathbb{R} = ker(|T(A_n)| - \sqrt{r_n}) \neq \{0\}$. Then $\mathbb{R}$ is a reducing subspace of $T(A_1), \dots, T(A_{n-1})$ and $(T(A_1)_{|\mathbb{R}}, \dots, (T(A_{n-1})_{|\mathbb{R}}$ is a doubly commuting $n-1$-tuple of posinormal operators on $\mathbb{R}$. Since $\sum_{i=1}^{n}\left(|T(A_i)| - \sqrt{r_i}\right)^2$ is not invertible,

then

$\ker(\sum_{i=1}^{n}(|T(A_i)| - \sqrt{r_i})^2) = \{\cap_{i=1}^{n-1} \ker(|A_i|) - \sqrt{r_i})\} \cap \mathbb{R} \neq \{0\}$.

Hence it follows that

$(\sqrt{r_1}, \dots, \sqrt{r_{n-1}}) \in \sigma(R)$,

where $R = (T(A_i)_{|\mathbb{R}}, \dots, T(A_{n-1})_{|\mathbb{R}})$.

Therefore by the induction hypothesis, there exists $(z_1, \dots, z_{n-1}) \in \sigma(S)$such that $|z_i| \geq \sqrt{r_i} : i = 1, \dots, n-1$,

where $S = (T(A_i)_{|\mathbb{R}}, \dots, T(A_{n-1})_{|\mathbb{R}})$. It thus follows that $(\overline{z_1}, \dots, \overline{z_{n-1}}) \in \sigma_p(S^*)$ there exists a non-zero vector x_0 in $\mathbb{R}$ such that $T(A^*)x_0 = \overline{z_i}x_0 : i = 1, \dots, n-1$.

Therefore

$\sum_{i=1}^{n-1}(T(A_i) - z_i)(T(A\_i) - z_i)^* + \left(|T(A_n) - \sqrt{r_n}|\right)^2$

is not invertible.

Hence

$\sum_{i=1}^{n-1}(T(A_i) - z_i)(T(A\_i) - z_i)^* + \left(|T(A_n) - \sqrt{r_n}|\right)^2 \neq 0$.

Let $\mathbb{P} = \ker(T(A_i) - z_i)(T(A\_i) - z\_i)^*$. Then $\mathbb{P}$ reduces $T(A_n)$. Also since $\mathbb{R} \cap \mathbb{P} \neq \{0\}$, $\sqrt{r_n} \in \sigma(|T(A_n)_{|N}|)$.

Since $T(A_n)_{|\mathbb{R}}$ is a posinormal operator then there is a $z_n \in \mathbb{C}$ such that $\left(T(A_n)_{|\mathbb{R}} - z_n\right)\left(T(A_n)_{|\mathbb{R}} - z_n\right)^*$ is not invertible and $|z_n|^2 \geq r_n$. Since $\sum_{i=1}^{n}(T(A_i) - z_i)(T(A\_i) - z_i)^*$ is not invertible, this point $z_1, \dots, z_n$ is in $\sigma(A)$ and satisfies $|z_i|^2 \geq r_i \quad (i = 1, \dots, n)$.

Thus the proof is complete.

Conclusions

The spectrum of a bounded posinormal operator A acting on a complex Hilbert space H satisfies Xia's property, i.e $\sigma(A) = \{z : \bar{z} \in \sigma_\pi(A^*)\}$. For a posinormal operator A if z is a member of the point spectrum of A the the closure of z is a member of the point spectrum of the adjoint of A. Lastly, doubly commuting n-tuples of posinormal operators are jointly normaloid.

Conflicts of Interest

Authors declare no conflict of interest.

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