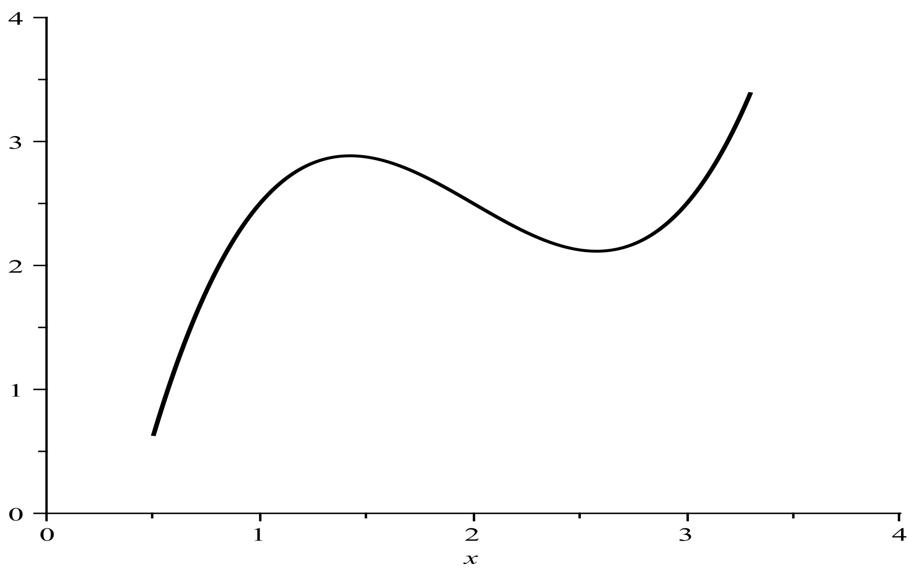
Calculus 3 - Surface Area

In calculus 1 we were able to find arc length using integrals. On a small interval, we create a small triangle. The hypotenuse approximates the length of the curve



If we denote dx, dy and ds and the lengths of each side then

$$ds^2 = dx^2 + dy^2 \tag{1}$$

or

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{2}$$

Now we add of the little line segments and in the limit, we obtain the integral

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{3}$$

If x and y are given parametrically $x = f(t), y = g(t)$, then this would become

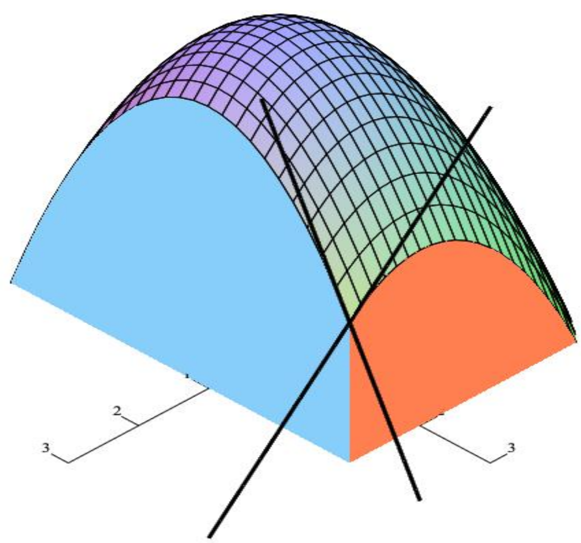
$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{4}$$

Surface Area

In 3D, the analogy to arc length is surface area. Recall when we obtained the tangent plane. We created two vectors

$$\vec{u} =< 1, 0, f_x >, \quad \vec{v} =< 0, 1, f_y >, \tag{5}$$

and evaluate these at some point (a, b).



We now cross these two vectors to get the normal so

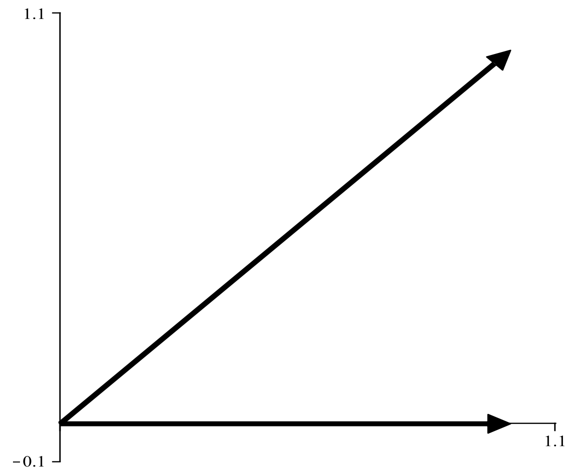
$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix} =< -f_x(a,b), -f_y(a,b), 1 > . \tag{6}$$

The equation of the tangent plane is then

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z-c) = 0 \tag{7}$$

where $c = f(a, b)$.

Let us return back to vectors from Calc 2. The area of the parallelogram



with $\|\vec{u}\|$ and $\|\vec{v}\|$ as sides is given by

$$A = \|\vec{u}\|\|\vec{v}\|\sin\theta \tag{8}$$

where θ is the angle between the vectors. It can be shown that

$$|\vec{u} \times \vec{v}| = \|\vec{u}\|\|\vec{v}\|\sin\theta \tag{9}$$

Now we create two small vectors

$$\vec{u} =< 1, 0, f_x > dx, \quad \vec{v} =< 0, 1, f_y > dy, \tag{10}$$

We now cross these two vectors to get the normal so

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} =< -f_x, -f_y, 1 > dxdy. \tag{11}$$

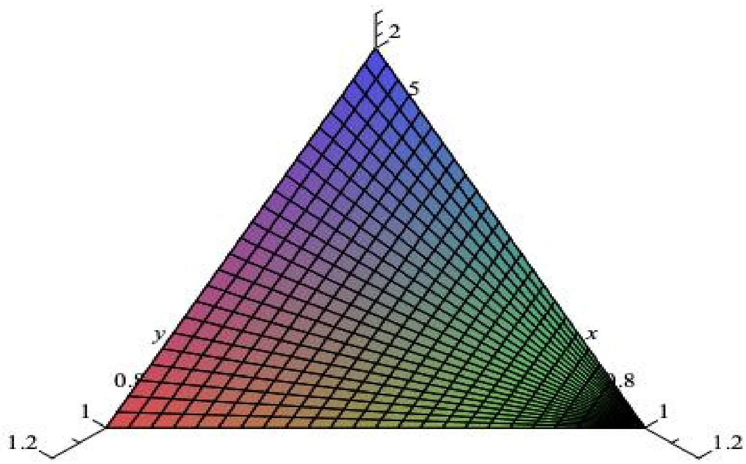
and then take the magnitude of this which gives

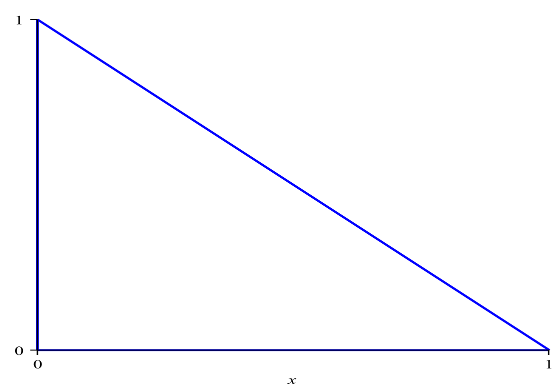
$$dSA = \sqrt{1 + f_x^2 + f_y^2}\, dxdy \tag{12}$$

Now we add up all the little areas and in the limit we obtain the double integral

$$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA. \tag{13}$$

Example 1. Find surface area of the plane of $2x + 2y + z = 2$ in the first octant.





Soln. We first find the partial derivatives so if $z = 2 - 2x - 2y$ then $f_x = -2, \quad f_y = -2$. The surface area is given by

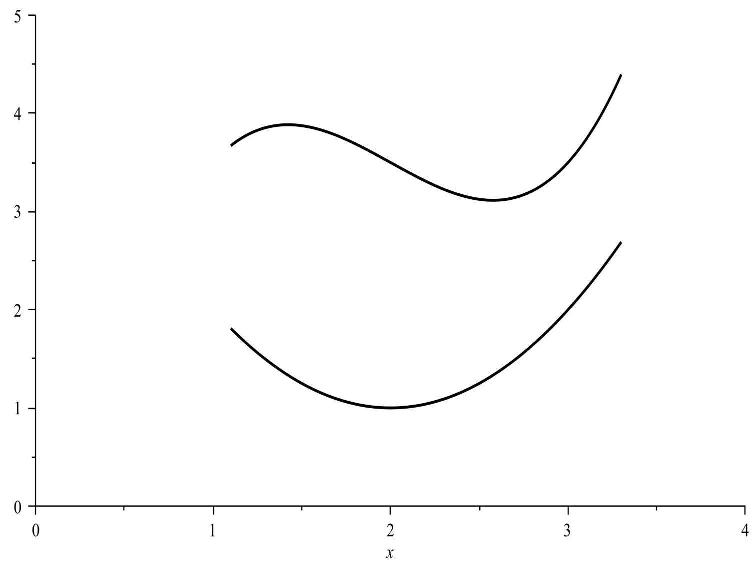
$$SA = \int_0^1 \int_0^{1-x} \sqrt{1 + 2^2 + 2^2}\, dydx = \frac{3}{2} \tag{14}$$

Area of Plane Regions

If the integrand is a number say 5. then

$$\iint\limits_R 5dA = 5A(R) \tag{15}$$

where $A(R)$ is the area of the region R. To show this consider



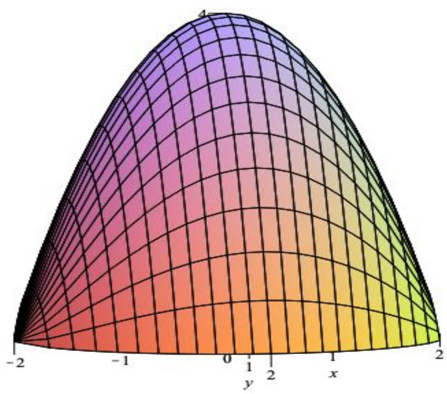
$$\int_a^b \int_{g(x)}^{h(x)} 1dydx = \int_a^b y\Big|_{g(x)}^{h(x)} dx = \int_a^b g(x) - h(x)dx \tag{16}$$

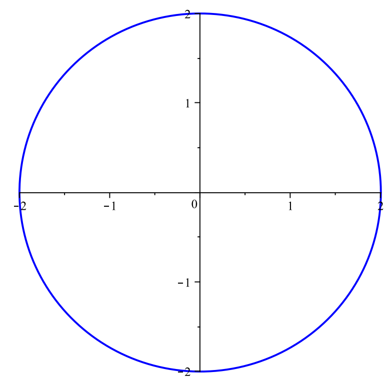
which is exactly the area of the region R.

Example 2. Find surface area of the paraboloid of $z = 4 - x^2 - y^2$ for $z \geq 0$

Soln. We first find the partial derivatives so

$$f_x = -2x, \quad f_y = -2y. \tag{17}$$





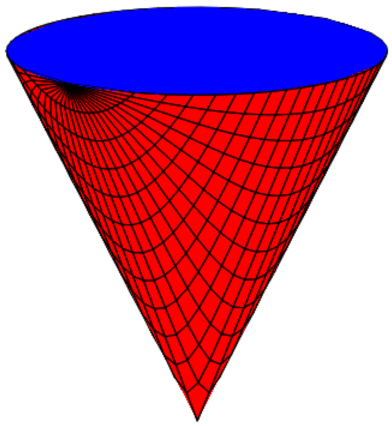
The surface area is given by

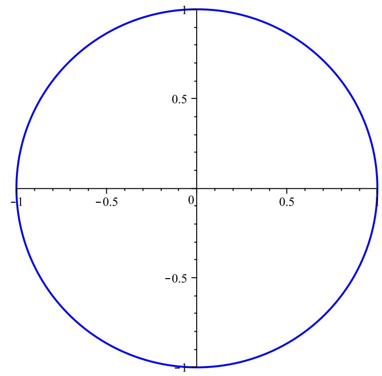
$$SA = \iint\limits_R \sqrt{1 + 4x^2 + 4y^2}\, dA \tag{18}$$

The region of integration is a circle of radius 2 so we switch to polar so

$$\begin{aligned} SA &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r dr d\theta \\ &= \int_0^{2\pi} \frac{1}{12} \left(1 + 4r^2\right)^{3/2} \Big|_0^2 d\theta \\ &= \frac{17\sqrt{17} - 1}{12} \int_0^{2\pi} d\theta \\ &= \frac{17\sqrt{17} - 1}{12} \cdot 2\pi \end{aligned} \tag{19}$$

Example 3. Find surface area of the cone of $z = \sqrt{x^2 + y^2}$ with the top of $z = 1$





Soln. We first find the partial derivatives so

$$f_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{y}{\sqrt{x^2 + y^2}}. \tag{20}$$

The surface area is given by

$$SA = \iint_R \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}}\, dA. \tag{21}$$

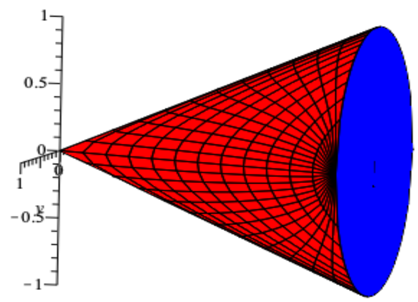
Simplifying the integrand gives

$$\sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{\frac{x^2 + y^2}{x^2 + y^2} + \frac{x^2 + y^2}{x^2 + y^2}} = \sqrt{2} \tag{22}$$

So the surface area of the outside of the cone is $\sqrt{2}$ times the area of the region R which is π so the surface area (including the top) is

$$SA = \sqrt{2}\pi + \pi \tag{23}$$

Example 4. Find surface area of the cone of $y = \sqrt{x^2 + z^2}$ with the top of $y = 1$



Soln. This is exactly the same problem as # 3 except the cone is on its side. We certainly could solve the equation of the cone for z but instead, let's use the variables x and z. The region of integration is still a circle but in the (x, z) plane.

In general, if the surface is given by $y = g(x, z)$ our surface area formula is

$$SA = \iint\limits_{R_{xz}} \sqrt{1 + g_x^2 + g_z^2}\, dA_{xz}. \tag{24}$$

where $dA_{xz} = dxdz$ or $dzdx$

Similarly, if the surface is given by $x = h(y, z)$ our surface area formula is

$$SA = \iint\limits_{R_{yz}} \sqrt{1 + h_y^2 + h_z^2}\, dA_{yz}. \tag{25}$$

where $dA_{yz} = dydz$ or $dzdy$