Calculus 3 - Multivariable Functions

In Calculus 1 (and even in high school) we introduced functions of a single variable. For example, $y = x^2$, $y = e^x$ or $y = \sin x$ and, in general $y = f(x)$. At first, we created a table of values and plotted a x vs y graph. In the case of say $y = x^2$ and simple table of values would be say

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

we plot the points and connect the points with a smooth curve.

We also asked questions like, for what values of x is the function defined (domain) and what are the attainable y values (range).

For example $y = \sqrt{1 - x^2}$. Here see see the graph is a half circle. The domain is

$$D = \left\{ x \,\middle|\, -1 \leq x \leq 1 \right\}$$

and the range

$$R = \left\{ y \,\middle|\, 0 \leq y \leq 1 \right\}.$$

So now we go to functions of more than one variable. Before doing so, one may ask why? Consider the following. You heat a rectangular plate at the center. As you move left and right or up and down the temperature of the
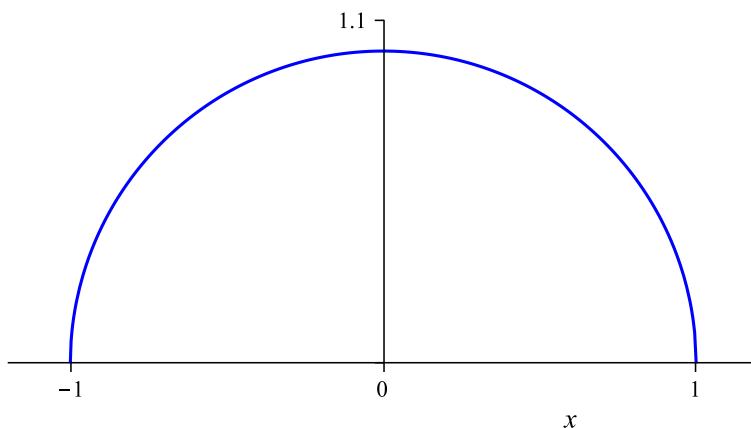
Figure 1: $y = \sqrt{1 - x^2}$

plate will change. If we denote left and right with the variable x and up and down as y, the the temperature T would be a function of both x and y and so

$$T = T(x, y).$$

This is just one of thousands of examples were we have functions of more than one variable. So we extend from $2D$ to $3D$.

$$y = f(x) \qquad\qquad z = f(x, y)$$

Table of Values

We certainly can do this. For example, if

$$z = x^2 + y^2$$

y,x	-3	-2	-1	0	1	2	3
-3	18	13	10	9	10	13	18
-2	13	8	5	4	5	8	13
-1	10	5	2	1	2	5	10
0	9	4	1	0	1	4	9
1	10	5	2	1	2	5	10
2	13	8	5	4	5	8	13
3	18	13	10	9	10	13	18

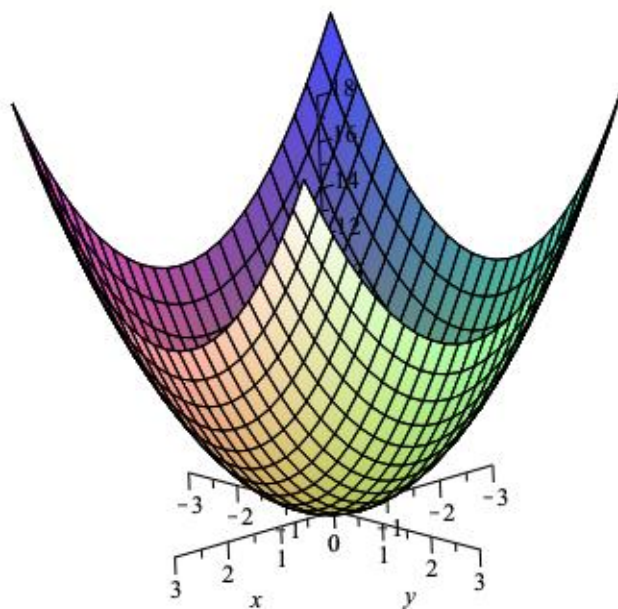which would give the following: but this usually takes to long and usually

Figure 2: $z = x^2 + y^2$

not very accurate. Computer algebra systems do a pretty job of drawing 3D graphs (I will use Maple a lot in this course and it was used to generate

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the plot is fig 2). Next week we will consider 10 very special surfaces called Quadratic Surfaces (these are in analogy to the quadratic curves in 2D (the line, parabola, ellipse and hyperbola).

Domain and Range

For what values of x and y is the function $z = f(x,y)$ defined and what are the attainable values z. Let us look at several examples.

Example 1 $z = x^2 + y^2$

From the graph (fig 2) we see that the domain is all (x,y) and the range is $z \geq 0$ or

$$D = \left\{ (x,y) \right\}$$
$$R = \left\{ z \mid z \geq 0 \right\}.$$

Example 2 $z = \sqrt{1 - x^2 - y^2}$

 From the graph (fig.3) we see that the domain is all (x,y) that are points inside the circle of radius 1 (boundary included) and the range is between 0 and 1 or

$$D = \left\{ (x,y) \mid 0 \leq x^2 + y^2 \leq 1 \right\}$$
$$R = \left\{ z \mid 0 \leq z \leq 1 \right\}.$$

Example 3 $z = \dfrac{1}{\sqrt{x^2 + y^2 - 1}}$

From the graph (fig.3) we see that the domain is all (x,y) that are points outside the circle of radius 1 (boundary excluded) and the range is $z \geq 0$
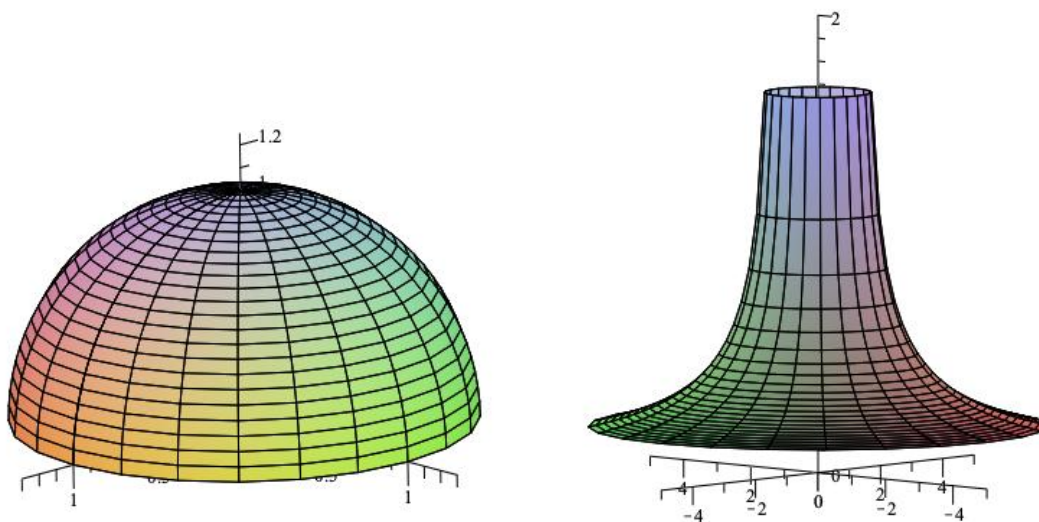
Figure 3: $z = \sqrt{1 - x^2 - y^2}$ and $z = \dfrac{1}{\sqrt{x^2 + y^2 - 1}}$

or

$$D = \left\{ (x, y) \,\middle|\, x^2 + y^2 > 1 \right\}$$
$$R = \left\{ z \,\middle|\, z > 0 \right\}.$$
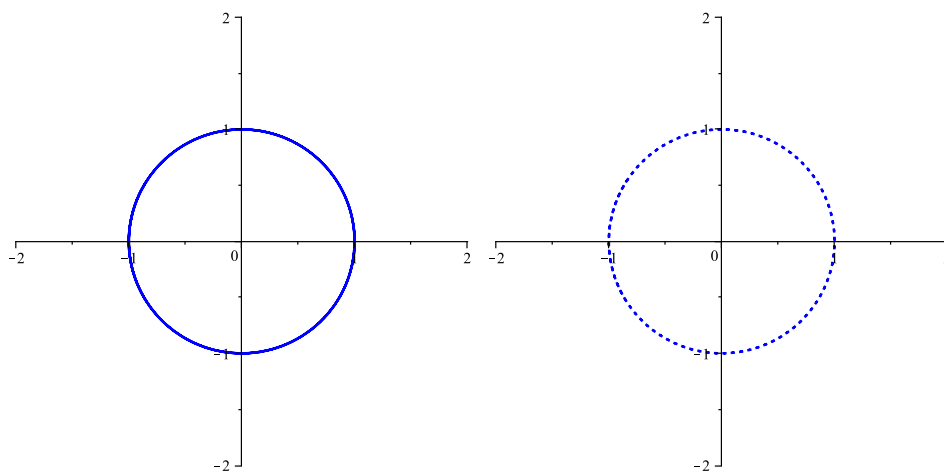
The domain's are as follows:

Figure 4: Domains of ex 2 and ex 3

Example 4 $z = \ln(y - x)$

For this example we need that $y - x > 0$ since the natural log is not even defined at zero and for negative numbers

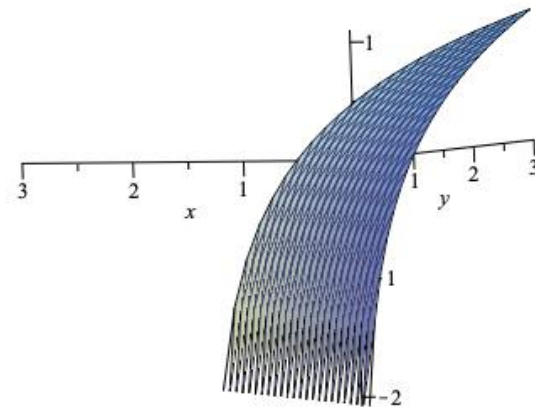$$D = \left\{ (x,y) \,\middle|\, y > x \right\}$$

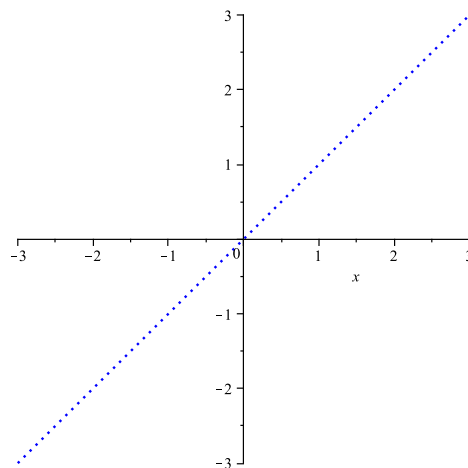Figure 5: $z = \ln(y - x)$

Figure 6: Domains of $z = \ln(y - x)$

Level Sets

If one looks at a map of the US and temperatures for a given day one might find lines on the map of equal temperature (isotherms). Given a function say $z = f(x,y)$ one could set the z value to a certain number(s) and then plot the $2D$ curves. These are referred to as *level sets* or *level curves*. An example is below where the function is $z = \exp(-x^2 \sin(2x)^2 - y^2)$
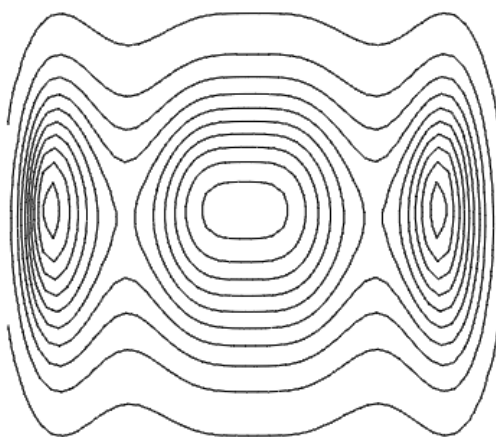

Figure 7: Level Sets

What's Next

Topics like: Limits, Continuity, Derivatives, Integrals that were introduced in Calc 1 also apply to functions of more than one variable. Then we move to the Calculus of what is called vector fields.