Calculus 3 - Vector Functions

In calculus 2 we introduced two ways of multiplying vectors: the dot product and cross product.

Dot Product The dot product of two vectors $\vec{u} =< u_1, u_2 >$ and $\vec{v} =< v_1, v_2 >$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

or in 3D where $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

The alternate definition is

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

where $\|\vec{u}\|$ and $\|\vec{v}\|$ is the magnitude of the two vectors and θ is the angle between the vectors.

Cross Product

Given vectors $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$ we define the cross product between two vectors as

$$\vec{u} \times \vec{v} =< u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 >$$

Now we define the cross product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors and c a scalar (a number)

$$\begin{aligned}
(i) &\quad \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \\
(ii) &\quad \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \\
(iii) &\quad c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) \\
(iv) &\quad \vec{0} \cdot \vec{v} = 0 \\
(v) &\quad \vec{u} \cdot \vec{u} = \|\vec{u}\|^2
\end{aligned}$$

Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors and c a scalar (a number)

$$\begin{aligned}
(i) &\quad \vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \\
(ii) &\quad \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w} \\
(iii) &\quad c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) \\
(iv) &\quad \vec{0} \times \vec{v} = \vec{0} \\
(v) &\quad \vec{u} \times \vec{u} = \vec{0} \\
(vi) &\quad \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}
\end{aligned}$$

With the introduction of derivative of vector functions

$$\vec{r}(t) = < f(t), g(t) > \quad \text{then} \quad \vec{r}\,'(t) = < f'(t), g'(t) > \tag{1}$$

We have the following derivative rules. Let $\vec{u}$ and $\vec{v}$ be vector functions and $f(t)$ a differentiable scalar function, and $\vec{c}$ a constant vector, then

$$
\begin{aligned}
(i) \quad & \frac{d}{dt}\vec{c} = \vec{0} \\
(ii) \quad & \frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \vec{u}'(t) + \vec{v}'(t) \\
(iii) \quad & \frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u} + f(t)\vec{u}'(t) \\
(iv) \quad & \frac{d}{dt}\left(\vec{u}(f(t))\right) = \vec{u}'(f(t))f'(t) \\
(v) \quad & \frac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t) \\
(vi) \quad & \frac{d}{dt}\left(\vec{u}(t) \times \vec{v}(t)\right) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)
\end{aligned}
$$

On thing that's important to realize is that if a vector has constant length (say c) then

$$\vec{u}(t) \cdot \vec{u}(t) = c^2 \tag{2}$$

and using the property (v) above then

$$\vec{u}'(t) \cdot \vec{u}(t) + \vec{u}(t) \cdot \vec{u}'(t) = 0 \tag{3}$$

so that

$$\vec{u}(t) \cdot \vec{u}'(t) = 0 \tag{4}$$

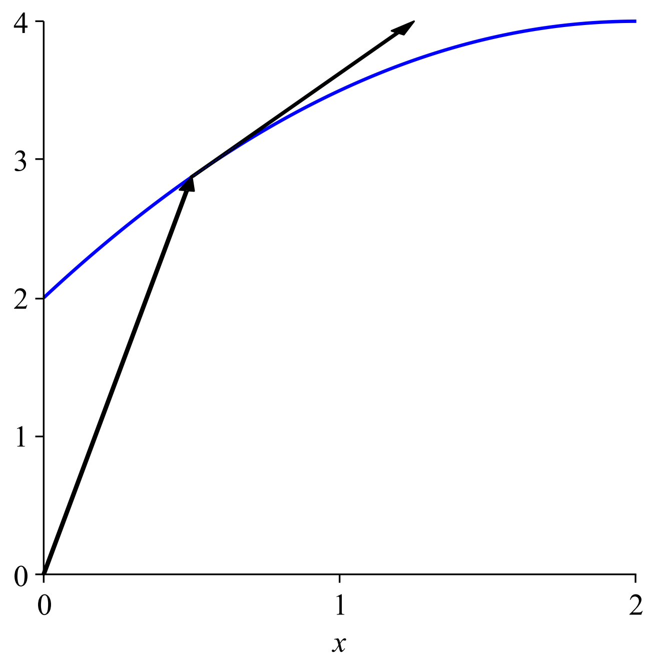
meaning that $\vec{u}(t)$ and $\vec{u}'(t)$ are perpendicular to one another. This is important in what follows.

Unit Tangent Vector

As we saw when we first introduced derivatives, that $\vec{r}'(t)$ is tangent to the space curve given by

$$x = f(t), \quad y = g(t). \tag{5}$$

so we define the unit tangent vector as



$$\vec{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \tag{6}$$

So, for example, if $\vec{r} =< t, \frac{1}{2}t^2 >$ then $\vec{r}' =< 1, t >$ and the unit tangent vector is

$$\vec{T} = \frac{< 1, t >}{\sqrt{1 + t^2}}, \tag{7}$$

if $\vec{r} =< \cos t, \sin t, t >$ then $\vec{r}' =< -\sin t, \cos t, 1 >$ and the unit tangent vector is

$$\vec{T} = \frac{< -\sin t, \cos t, 1 >}{\sqrt{2}} \tag{8}$$

since the magnitude of $\vec{r}'(t)$ is $\sqrt{2}$. Since the tangent vector is a unit vector, then the derivative of this would give another vector that is perpendicular to $\vec{T}'$.

Unit Normal Vector

We define the unit normal vector as

$$\vec{N} = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \tag{9}$$

If we consider the examples above then

$$\vec{T}' = \left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle \tag{10}$$

then

$$\|\vec{T}'\| = \sqrt{\left(\frac{-t}{(1+t^2)^{3/2}}\right)^2 + \left(\frac{1}{(1+t^2)^{3/2}}\right)^2} = \frac{1}{1+t^2} \tag{11}$$

and we obtain

$$\vec{N} = \frac{\left\langle \frac{-t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle}{\frac{1}{1+t^2}} = \left\langle \frac{-t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}} \right\rangle \tag{12}$$

and the reader can verify that $|\vec{T}| = 1$, $|\vec{N}| = 1$ and $\vec{T} \cdot \vec{N} = 0$.

In the second example, where

$$\vec{T} = \frac{< -\sin t, \cos t, 1 >}{\sqrt{2}} \tag{13}$$

then

$$\vec{T}' = \frac{< -\cos t, -\sin t, 0 >}{\sqrt{2}} \tag{14}$$

$|\vec{T}'| = \dfrac{1}{\sqrt{2}}$ and $\vec{N}$ is given by

$$\vec{N} =< -\cos t, -\sin t, 0 > \tag{15}$$

and the reader can also verify that $\|\vec{T}\| = 1$, $\|\vec{N}\| = 1$ and $\vec{T} \cdot \vec{N} = 0$.

Unit Binormal

If we are in 3D, we now have two vectors, the unit tangent vector and the unit normal vector. We can create a third vector which is perpendicular to both. We define the unit Binormal as

$$\vec{B} = \vec{T} \times \vec{N} \tag{16}$$

For the example above where $\vec{r} =< \cos t, \sin t, t >$ then

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} =< \frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$$