Math 2371 Calc III
Sample Test 3 - Solns

1.(i) Is the following vector field conservative?

$$\vec{F} = < yz + 3, xz + 4y, xy + 3z^2 > .$$

If so, find the potential ϕ. Use this to evaluate

$$\int_c (yz+3)dx + (xz+4y)dy + (xy+3z^2)dz$$

where c is any path from $(0,0,0)$ to $(1,2,3)$.
Soln. Since $\nabla \times \vec{F} = 0$ then yes, the vector field is conservative. Thus f exists such that $\vec{F} = \vec{\nabla} f$ so

$$\begin{aligned} f_x &= yz + 3 &&\Rightarrow \quad f = x^2y + A(y,z) \\ f_y &= xz + 4y &&\Rightarrow \quad f = x^2y + yz^2 + B(x,z) \\ f_z &= xy + 3z^2 &&\Rightarrow \quad f = yz^2 + C(x,y) \end{aligned}$$

Therefore we see that

$$f = xyz + 3x + 2y^2 + z^3 + c.$$

(b) Evaluate the following where c is any path from $(0,0,0)$ to $(1,2,3)$.

$$\int_c (yz+3)dx + (xz+4y)dy + (xy+3z^2)dz$$

Soln.

$$\int_C (yz+3)dx + (xz+4y)dy + (xy+3z^2)dz = \left. xyz + 3x + 2y^2 + z^3 \right|_{(0,0,0)}^{(1,2,3)} = 44$$

1. (ii) Is the following vector field conservative? If so, find the potential ϕ

$$\vec{F} = < 2xy, x^2 + z^2, 2yz > .$$

If so, find the potential ϕ. Use this to evaluate

$$\int_c 2xydx + (x^2+z^2)dy + 2yzdz$$

where c is any path from $(0,0,0)$ to $(1,2,3)$.
Soln. Since $\nabla \times \vec{F} = 0$ then yes, the vector field is conservative. Thus f exists such that $\vec{F} = \vec{\nabla} f$ so

$$\begin{aligned} f_x &= 2xy &&\Rightarrow \quad f = x^2y + A(y,z) \\ f_y &= x^2 + z^2 &&\Rightarrow \quad f = x^2y + yz^2 + B(x,z) \\ f_z &= 2yz &&\Rightarrow \quad f = yz^2 + C(x,y) \end{aligned}$$

Therefore we see that

$$f = x^2y + yz^2 + c.$$

(b) Evaluate the following where c is any path from $(0,0,0)$ to $(1,2,3)$.

$$\int_c 2xydx + (x^2 + z^2)dy + 2yzdz$$

Soln.

$$\int_c 2xydx + (x^2 + z^2)dy + 2yzdz = x^2y + yz^2\Big|_{(0,0,0)}^{(1,2,3)} = 20$$

2. Evaluate the following line integral $\int_c xy\,ds$ where c is counterclockwise direction around a circle of radius 1 from $(1,0)$ to $(0,1)$.

Soln. Here parameterize the circle of radius $r = 1$ with $x = \cos t, y = \sin t$. Now

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t \tag{1.1}$$

so

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \sqrt{\sin^2 t + \cos^2 t}dt = dt \tag{1.2}$$

To evaluate the integral is to evaluate

$$\int_0^{\pi/2} \cos t \sin t dt = \frac{1}{2}\sin^2 t\Big|_0^{\pi/2} = \frac{1}{2} \tag{1.3}$$

3. Green's Theorem is

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Verify Green's Theorem where $\vec{F} =< y^2, x^2 + 2xy >$ where R is the region bound by the curves $y = x^2$, $y = 1$ and $x = 0$ in $Q1$.

Soln. Again, we have three separate curves which we denote by C_1, C_2 and C_3.

C_1 : Here $y = x^2, dy = 2x\,dx$ so $\int_0^1 x^4dx + (x^2 + 2x^3)2x\,dx = 3/2$

C_2 : Here $y = 1, dy = 0$ so $\int_1^0 dx = -1$

C_3 : Here $x = 0, dx = 0$ so $\int_{C_3} 0 = 0$

Thus $\int_c y^2dx + (x^2 + 2xy)dy = 3/2 - 1 + 0 = 1/2$.

Since $P = y^2$ and $Q = x^2 + 2xy$ then $Q_x - P_y = 2x + 2y - 2y = 2x$ so

$$\iint_R (Q_x - P_y)\,dA = \int_0^1 \int_{x^2}^1 2xdy\,dx = 1/2$$

4. Evaluate $\iint_S z\,dS$ where S is the surface of the paraboloid $z = 1 - x^2 - y^2, z \geq 0$.

Soln. Since $z = 1 - x^2 - y^2$ then $dS = \sqrt{1 + z_x^2 + z_y^2}\,dA = \sqrt{1 + 4x^2 + 4y^2}dA$ and so far we have $\iint_R \left(1 - x^2 - y^2\right) \sqrt{1 + 4x^2 + 4y^2}\,dA$ where the region of integration is the circle $x^2 + y^2 = 1$. Switching to polar gives

$$\int_0^{2\pi} \int_0^1 \left(1 - r^2\right) \sqrt{1 + 4r^2} r dr d\theta = \left(\frac{5\sqrt{5}}{24} - \frac{11}{120}\right) 2\pi$$

5. Find the flux $\iint_S \vec{F} \cdot \hat{n} dS$ of the vector field $\vec{F} =< 2x, 2y, 2z + 2 >$ through the surface of the plane $x + y + z = 1$ in the first quadrant.

Soln. The unit normal to the surface is given by $\vec{n} = \dfrac{< 1, 1, 1 >}{\sqrt{3}}$. For this surface $dS = \sqrt{1 + 1 + 1}\,dA$ so

$$\begin{aligned} \vec{F} \cdot \vec{n} dS &= \iint_S < 2x, 2y, 2z + 2 > \cdot \frac{< 1, 1, 1 >}{\sqrt{3}} \sqrt{1 + 1 + 1}\,dA \\ &= \iint_S (2x + 2y + 2z + 2)\,dA \end{aligned}$$

Bringing in the surface we obtain

$$\int_0^1 \int_0^{1-x} 4\,dy\,dx = 2$$