Math 2471 – Calc III

Divergence Thm

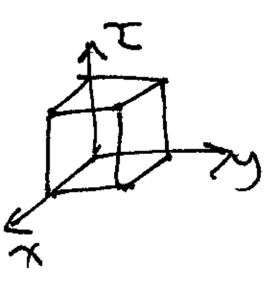
Let $\bar{F}$ be a vector field whose components have continuous 1^{st} derivatives in a simple connected region D in $\mathbb{R}^3$ enclosed by an oriented surface S.

Then $$\iint_S \bar{F}\cdot\hat{n}\,dS = \iiint_V \bar{\nabla}\cdot\bar{F}\,dV$$

Ex 1 Verify Div Thm when

$$\bar{F} = \langle 2x, -2y, z^2\rangle$$

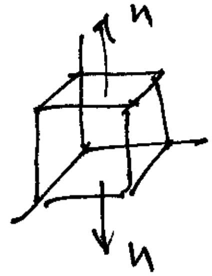
& S is the cube bound by the plane $x=0, x=1, y=0, y=1, z=0, z=1$



1st the surface integrals. There are 6 surface

S_1 : top

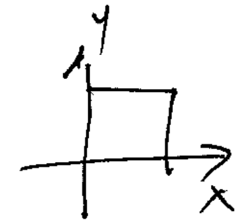
S_2 : bottom



$z=1, \quad \hat{n} = \langle 0,0,1\rangle$

$\bar{F} = \langle 2x, -2y, 1\rangle$

so $\bar{F}\cdot\hat{n} = 1$ so $\int_0^1\int_0^1 1\,dy\,dx = 1$

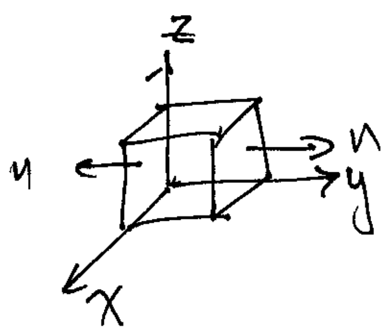


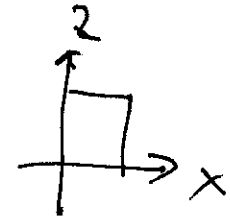
S_2: $z=0 \quad \hat{n} = \langle 0,0,-1\rangle$ so $\bar{F} = \langle 2x,-2y,0\rangle$

$\bar{F}\cdot\hat{n} = 0$ so no flux

S_3 : right

S_4 : left





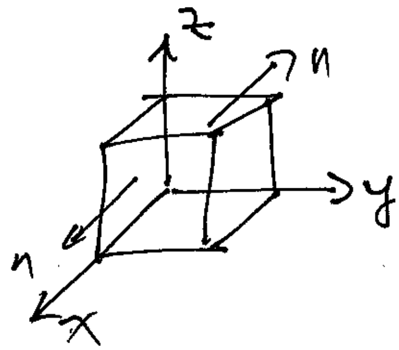
$S_3 \quad y = 1 \quad \hat{n} = \langle 0,1,0\rangle \quad \bar{F} = \langle 2x,-2,z^2\rangle$

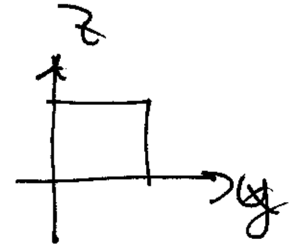
so $\bar{F}\cdot\hat{n} = -2 \quad \int_0^1\int_0^1 -2\,dx\,dz = -2$

$S_4 \quad y=0 \quad \hat{n} = \langle 0,-1,0\rangle \quad \bar{F} = \langle 2x,0,z^2\rangle$

so $\bar{F}\cdot\hat{n} = 0$ so no flux

S_5 front
S_6 back





S_5 $\hat{n} = \langle 1, 0, 0 \rangle$

$x = 1$ so $\bar{F} = \langle 2\cancel{x}, -2y, z^2 \rangle$

$\bar{F} \cdot \hat{n} = 2$ so $\int_0^1 \int_0^1 2 \, dz \, dy = 2$

S_6 $\hat{n} = \langle -1, 0, 0 \rangle$ $x = 0$ so $\bar{F} = \langle 0, -2y, z^2 \rangle$

$\bar{F} \cdot \hat{n} = 0$ no flux

Now add up flux's

$1 + 0 - 2 + 0 + 2 + 0 = 1$ //

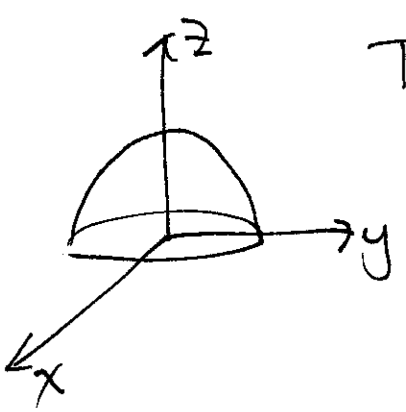
2b part $\text{Div } \bar{F} = 2 - 2 + 1 = 1$

$\int_0^1 \int_0^1 \int_0^1 1 \, dz \, dy \, dx = 1$ // Same.

Ex 2 $\quad \vec{F} = \langle xz, yz, 2z^2 \rangle$

where S is the surface bound by

$z = 1 - x^2 - y^2$ & $z = 0$



There are 2 surfaces here the paraboloid & the bottom

S1 Paraboloid $\quad z = 1 - x^2 - y^2$

Now $\quad \vec{n} = \langle -g_x, -g_y, 1 \rangle = \langle 2x, 2y, 1 \rangle$

Q: is this an outward normal - Yes!

$$\hat{n} = \frac{\langle 2x, 2y, 1 \rangle}{\sqrt{1 + 4x^2 + 4y^2}}, \quad dS = \sqrt{1 + 4x^2 + 4y^2}\, dA$$

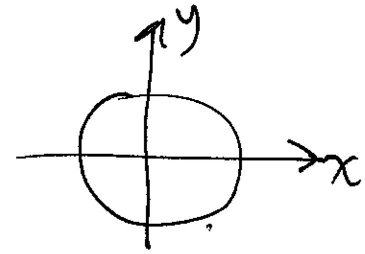
so $\quad \vec{F} \cdot \hat{n}\, dS = \langle xz, yz, 2z^2 \rangle \cdot \frac{\langle 2x, 2y, 1 \rangle}{\cancel{\sqrt{1 + 4x^2 + 4y^2}}} \cancel{\sqrt{1 + 4x^2 + 4y^2}}\, dA$

$= 2x^2 z + 2y^2 z + 2z^2$

Now bring in surface

so $\bar{F}\cdot\hat{n}\,dS = (2x^2+y^2)(1-x^2-y^2)+2(1-x^2-y^2)^2$

the region of integration is the circle of radius 1



so $\displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left[2r^2(1-r^2)+2(1-r^2)^2\right] r\,dr\,d\theta$

$\displaystyle 2\int_0^{2\pi}\int_0^1 \left(r^2-\cancel{r^4}+1-2r^2+\cancel{r^4}\right)\cdot r\,dr\,d\theta$

$\displaystyle 2\int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta = 2\int_0^{2\pi} \frac{r^2}{2}-\frac{r^4}{4}\Big|_0^1 d\theta$

$\displaystyle = 2\int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{1}{2}\theta\Big|_0^{2\pi} = \pi$

second surface circle where $z=0$

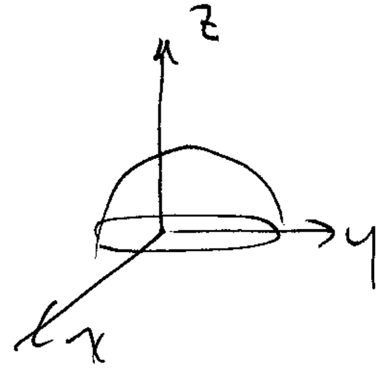
so $\bar{F}=\langle 0,0,0\rangle$ so $\bar{F}\cdot\hat{n}=0$ so no flux

so $\displaystyle\iint_S F\cdot\hat{n}\,dS = \pi$ //

2nd part

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial}{\partial x}xz + \frac{\partial}{\partial y}yz + \frac{\partial}{\partial z}2z^2 = z+z+4z$$
$$= 6z$$

so $$\iiint_V \bar{\nabla}\cdot\bar{F}\,dV$$



$$= 6\iiint_V z\,dV$$

Here switch to cylindrical polar's

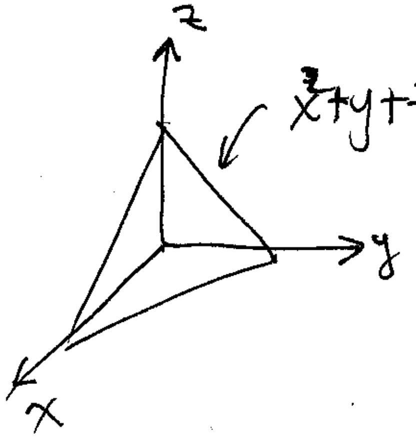
$$6\int_0^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1-r^2} z\,r\,dz\,dr\,d\theta$$

$$6\int_0^{2\pi}\int_0^1 \frac{rz^2}{2}\Big|_0^{1-r^2} dr\,d\theta = \frac{6}{2}\int_0^{2\pi}\int_0^1 r(1-r^2)^2\,dr\,d\theta$$

$$= \frac{6}{2}\int_0^{2\pi}\frac{1}{6}\,d\theta = \frac{1}{2}\int_0^{2\pi}d\theta = \frac{2\pi}{2} = \pi$$

same answer

ex 3



$\bar{F} = \langle x^2, 2yz, z^2 \rangle$

surfaces 1st

S_1 bottom $\hat{n} = \langle 0,0,-1 \rangle$ $z=0$ so $\bar{F} = \langle x^2, 0, 0 \rangle$

so $\bar{F} \cdot \hat{n} = 0$ no flux

S_2 back $\hat{n} = \langle -1, 0, 0 \rangle$ $x=0$ $\bar{F} = \langle 0, 2yz, z^2 \rangle$

so $\bar{F} \cdot \hat{n} = 0$ no flux

S_3 left $\hat{n} = \langle 0,-1,0 \rangle$ $y=0$ $\bar{F} = \langle x^2, 0, z^2 \rangle$

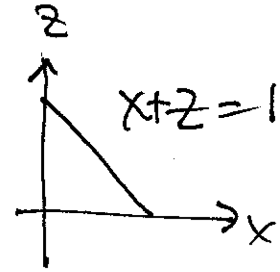
so $\bar{F} \cdot \hat{n} = 0$ no flux

S_4 plan $\hat{n} = \dfrac{\langle 1,1,1 \rangle}{\sqrt{3}}$ $\bar{F} \cdot \hat{n} = \langle x^2, 2yz, z^2 \rangle \cdot \dfrac{\langle 1,1,1 \rangle}{\sqrt{3}}$

$dS = \sqrt{3}\, dA$

$$\iint_{R_{xy}} \frac{x^2+2yz+z^2}{\cancel{\sqrt{3}}} \cancel{\sqrt{3}}\, dA = \iint_R x^2+2yz+z^2 \, dA$$

I would solve surface for $y = 1 - x - z$



$$\int_0^1 \int_0^{1-x} x^2 + 2(1-x-z)z + z^2 \, dz\,dx$$

$$\int_0^1 \int_0^{1-x} (x^2 + 2z - 2xz - 2z^2 + z^2)\, dz\,dx$$

$$\int_0^1 x^2 z + z^2 - xz^2 - \frac{z^3}{3} \Big|_0^{1-x} dx$$

$$\int_0^1 x^2(1-x) + (1-x)^2 - x(1-x)^2 - \left(\frac{1-x}{3}\right)^3 dx$$

$$\int_0^1 \left(3x^2 - \frac{5}{3}x^3 + \frac{2}{3} - 2x\right) dx = \frac{1}{4}$$

so total flux $= 0 + 0 + 0 + \frac{1}{4}$

$$= \frac{1}{4}$$

Now to the second part

$$\bar{\nabla} \cdot \bar{F} = \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(z^2) = 2x + 2z + 2z$$
$$= 2x + 4z$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (2x+4z)\,dz\,dy\,dx$$

$$\int_0^1 \int_0^{1-x} 2xz + 2z^2 \Big|_0^{1-x-y} dy\,dx$$

$$\int_0^1 \int_0^{1-x} 2x(1-x-y) + 2(1-x-y)^2\,dy\,dx$$

$$\int_0^1 \int_0^{1-x} (-2x + 2xy + 2 - 4y + 2y^2)\,dy\,dx$$

$$\int_0^1 -2xy + xy^2 + 2y - 2y^2 + \frac{2y^3}{3} \Big|_0^{1-x} dx$$

$$\int_0^1 -2x(1-x) + x(1-x)^2 + 2(1-x) - 2(1-x)^2 + \frac{2}{3}(1-x)^3\,dx$$

$$\int_0^1 \left(\frac{2}{3} - x + \frac{1}{3}x^3\right) dx = \frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{12} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{2} + \frac{1}{12} = \frac{8-6+1}{12} = \frac{3}{12} = \frac{1}{4} \,//\, \text{same}$$