VANDERBILT UNIVERSITY School of Engineering

Discrete Structures

CS 2212

(Fall 2020)

16 – Induction and Recurrence

Reminders and Recap

Previously:

Sequences

Summations

Today:

Closed form of Summations

- Substitution method
- Cancellation method

Induction

Summations and Closed Forms

Last time (Summary):

Given a sequence (**terms**): $a_1, a_2, a_3, \ldots a_n$

Find sequence sum: $a_1 + a_2 + a_3 + \ldots + a_n$

$a_1 + a_2 + a_3 + \ldots + a_n$ [Expanded form]

Find pattern – i^{th} term (a_i)

$$\sum_{i=1}^{n} a_i$$

[Summation]

Simplify

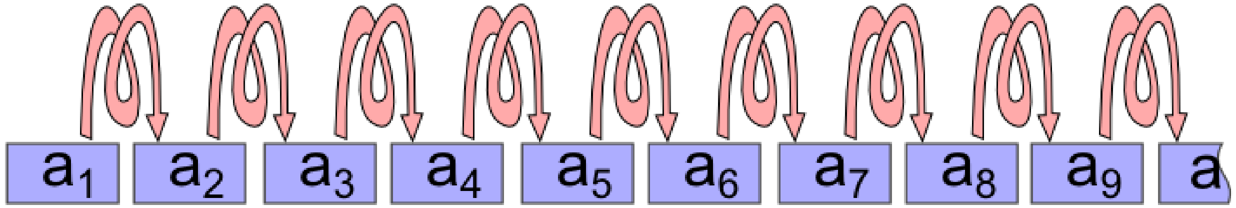
Closed form in terms of n [Closed form]

Summations and Recurrence Relation

Next:

What if sequence is given in the form of a **recurrence relation** (instead of individual terms) ?

$$a(n) = c_1 a(n-1) + c_2$$



How can we find the closed form of the recurrence relation $a(n)$?

Summations and Closed Forms

Approach:

$a(n) = c_1\, a(n-1) + c_2$

$$a(n) = \sum_{i=1}^{n} (\text{in terms of } i)$$

Closed form in terms of n

[Recurrence relation]

Backward Substitution

Find pattern – i^{th} term

[Summation form]

Simplify

[Closed form]

Recurrence Relations: Substitution Method

Given a recurrence relation:

$$a(n) = a(n\text{-}1) + 2n; \qquad a(0) = 1$$

Find the closed form of $a(n)$.

Step 1:

Work **backwards** to *unravel* and write out some terms.

$$a(n) = a(n-1) + 2n$$

$$a(n-1) = a(n-2) + 2(n-1)$$

$$a(n-2) = a(n-3) + 2(n-2)$$

Recurrence Relations: Substitution Method

Step 1: Work **backwards** to *unravel* and write out some terms.

$$a(n) = a(n-1) + 2n$$

$$a(n-1) = a(n-2) + 2(n-1)$$

$$a(n-2) = a(n-3) + 2(n-2)$$

Step 2: Use **substitutions** and plug in for the terms.

$$a(n) = a(n-1) + 2n$$

$$a(n) = a(n-2) + 2(n-1) + 2n$$

Recurrence Relations: Substitution Method

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Recurrence Relations: Substitution Method

Step 3: Look for a **pattern** among the terms and visualize the terms over its entirety.

$$a(n) = a(n-3) + 2(n-2) + 2(n-1) + 2n$$

i^{th} term = $\boldsymbol{2\,i}$

Step 4: Establish the summation formula and then determine the closed form

$$\sum_{i=1}^{n}(2i) = 2\left(\frac{n(n+1)}{2}\right) = n^2 + n$$

So, the final closed form would be: $\boldsymbol{n^2 + n + 1} = O(n^2)$

Add 1 for the initial term $a(0)$

Substitution Method – More Examples

Practice: Analyze how much work a recursive algorithm does by precisely calculating its closed form based on the following recurrence relation.

$$r(n) = 2r(n-1) + 1; \quad r(0) = 1$$

Recurrence Relations: Substitution Method

Step 1: Work backwards and write out some terms:

$$r(n) = 2r(n-1) + 1$$

$$r(n-1) = 2r(n-2) + 1$$

$$r(n-2) = 2r(n-3) + 1$$

$$r(n-3) = 2r(n-4) + 1$$

Step 2: Use substitutions for the terms…

$$r(n) = 2\,r(n-1) + 1$$

$$= 2\,[2r(n-2) + 1] + 1$$

$$= 2^2\,r(n-2) + 2 + 1$$

Recurrence Relations: Substitution Method

Step 2 (con't):

$$r(n) = 2\,r(n-1) + 1$$

$$= 2\,[\,2r(n-2) + 1\,] + 1$$

Recurrence Relations: Substitution Method

Step 2 (con't):

$$r(n) = 2\,r(n-1) + 1$$

$$= 2\,[\,2r(n-2) + 1\,] + 1$$

$$= 2^2\,r(n-2) + 2 + 1$$

$$= 2^2\,[2r(n-3) + 1] + 2 + 1$$

$$= 2^3\,r(n-3) + 2^2 + 2 + 1$$

Do you see a pattern? This looks like a geometric sequence (there's a summation formula for that).

$$2^0 + 2^1 + 2^2 + 2^3 + \ldots + 2^n$$

Recurrence Relations: Substitution Method

Step 4: Try to establish the summation formula and determine the closed form.

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}; a \neq 1$$

This leads us to a closed form of $2^{n+1} - 1$ which is **$O(2^n)$**.

Substitution Method – More Examples

Practice: Analyze how much work a recursive algorithm does by precisely calculating its closed form based on the following recurrence relation.

$$g(n) = g(n\text{-}1) + 2n - 1; \qquad g(0) = 0$$

Substitution Method – More Examples

Step 1: Work backwards and write out some terms:

$$g(n) = g(n\text{-}1) + 2n - 1$$
$$g(n\text{-}1) = g(n\text{-}2) + 2(n\text{-}1) - 1$$
$$g(n\text{-}2) = g(n\text{-}3) + 2(n\text{-}2) - 1$$

Step 2: Use substitutions for the terms…

$$g(n) = g(n\text{-}1) + 2n - 1$$
$$g(n) = [\, g(n\text{-}2) + 2(n\text{-}1) - 1] + 2n - 1$$

Substitution Method – More Examples

Step 1: Work backwards and write out some terms:

$$g(n) = g(n\text{-}1) + 2n - 1$$

$$g(n\text{-}1) = g(n\text{-}2) + 2(n\text{-}1) - 1$$

$$g(n\text{-}2) = g(n\text{-}3) + 2(n\text{-}2) - 1$$

Step 2: Use substitutions for the terms…

$$g(n) = g(n\text{-}1) + 2n - 1$$

$$g(n) = [g(n\text{-}2) + 2(n\text{-}1) - 1] + 2n - 1$$

$$g(n) = [g(n\text{-}3) + 2(n\text{-}2) - 1] + 2(n\text{-}1) - 1 + 2n - 1$$

Substitution Method – More Examples

Step 3: Look for a pattern among the terms and visualize the terms over its entirety.

(Tip: *It often helps to reorganize and simplify a bit first.***)**

$$g(n) = g(n\text{-}3) + 2(n\text{-}2) - 1 + 2(n\text{-}1) - 1 + 2n - 1$$

Group the **1**'s together.

$$g(n) = g(n\text{-}3) + 2(n\text{-}2) + 2(n\text{-}1) + 2n - \mathbf{3}$$

Let's now visualize the above pattern over all the terms in the recursion and see what that looks like

Substitution Method – More Examples

(Tip: *Sometimes it helps to think backwards.***)**

$g(n)$

$= \quad 2(n-(n-1)) \; + \; 2(n-(n-2)) \; + \ldots + \; 2(n-2) \; + \; 2(n-1) \; + \; 2n \; - \; n$

$= \quad 2(1) \; + \; 2(2) \; + \ldots + \; 2(n-2) \; + \; 2(n-1) \; + \; 2n \; - \; n$

Step 4: Find the summation formula and then closed form

Do you see the summation pattern? It looks like this…

$$\sum_{k=1}^{n} (2k) - n$$

Substitution Method – More Examples

Use your summation rules to help derive the closed form.

$$\sum_{k=1}^{n}(2k) - n$$

$$= 2\sum_{k=1}^{n}(k) - n$$

$$= 2\left(\frac{n(n+1)}{2}\right) - n = n^2 + n - n = n^2$$

We conclude this is an **O(n²)** recursive algorithm.

Recurrence Relations: Cancellation Method

Another technique that is useful for solving a recurrence relation is the:

Cancellation method.

Summations and Closed Forms

Approach:

$$a(n) = c_1\, a(n-1) + c_2$$

$$a(n) = \sum_{i=1}^{n} (\text{in terms of } i)$$

Closed form in terms of n

[Recurrence relation]

Backward Substitution

Find pattern – i^{th} term

[Summation form]

Simplify

[Closed form]

Summations and Closed Forms

Approach:

$a(n) = c_1\, a(n\text{-}1) + c_2$

$$a(n) = \sum_{i=1}^{n} \left(\text{in terms of } i\right)$$

Closed form in terms of n

[Recurrence relation]

Cancellation Method

Find pattern – i^{th} term

[Summation form]

Simplify

[Closed form]

Recurrence Relations: Cancellation Method

How it works:

Same as the substitution method

1. Start with the general (recurrence) equation **r(n)** and continue writing new equations whose left side is the term involving the recurrence **r** on the right side of the **previous equation**.

2. Look for a pattern that allows us to skip ahead and write the last equation **r(0)** on the right side.

3. **Add** up the equations and **cancel** the like terms to obtain an expression for **r(n)**.

Recurrence Relations: Cancellation Method

Example: Solve the following recurrence using cancellation:

$a(0) = 1$

$a(n) = a(n - 1) + 2n$

Step 1: As with substitution, we start by writing out some terms:

$$a(n) = a(n - 1) + 2n$$

$$a(n-1) = a(n - 2) + 2(n - 1)$$

$$a(n-2) = a(n - 3) + 2(n - 2)$$

…

$a(n-(n-1)) = a(0) + 2(n-(n-1))$ same as $a(1) = a(0) + 2$

Recurrence Relations: Cancellation Method

Step 2: Next we add up the terms on both sides of the = sign

$$a(n) + \cancel{a(n-1)} + \cancel{a(n-2)} + \cancel{a(n-3)} + \ldots + \cancel{a(1)} + \cancel{a(0)}$$

$$=$$

$$\cancel{a(n-1)} + 2n + \cancel{a(n-2)} + 2(n-1) + \cancel{a(n-3)} + 2(n-2)$$
$$+ \ldots + a(0) + 2 + \cancel{a(0)}$$

Step 3: Cancel the like terms to obtain the equation for **a(n)**.

$$a(n) = a(0) + 2(1) + \ldots + 2(n-2) + 2(n-1) + 2n$$

Recurrence Relations: Cancellation Method

Step 4: Determine the **summation** and closed form.

$$a(n) = a(0) + 2(1) + \dots + 2(n-2) + 2(n-1) + 2n$$

$$a(n) = 1 + 2\,[\,1 + \dots + (n-2) + (n-1) + n\,]$$

$$a(n) = 1 + 2\left[\frac{n(n+1)}{2}\right]$$

$$a(n) = n^2 + n + 1$$

This is an $O(n^2)$ algorithm

Mathematical Induction

Mathematical Induction

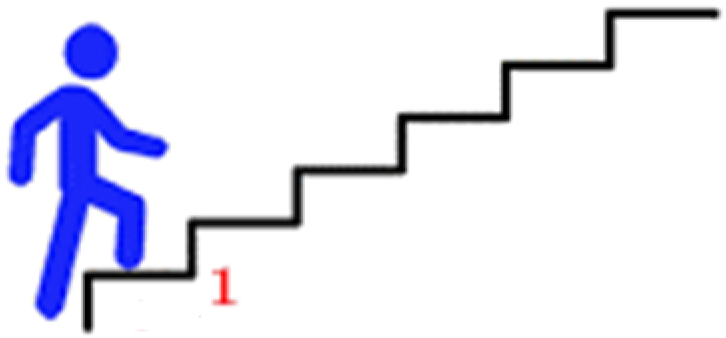
General Idea of Induction:

Show that a person can provably climb to any arbitrary step.

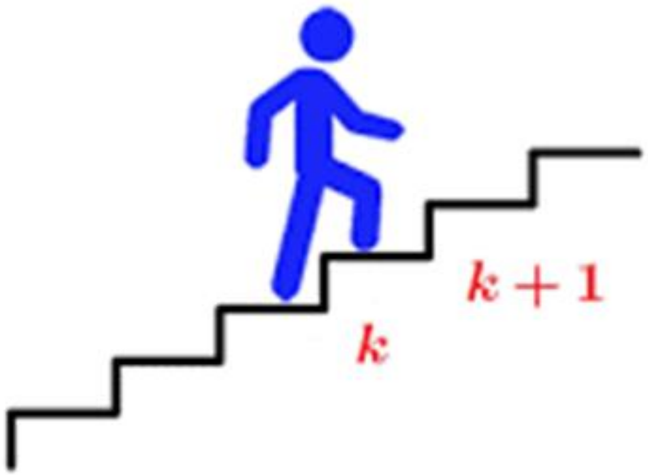
For this, all we need to show is

1) A person can climb the first step.

2) A person knows to climb from any step k to the next step $k+1$.



Part 1



Part 2

Induction – A powerful proof technique

An inductive proof establishes that some statement $P(n)$ parameterized by n is true, for any positive n.

Inductive proofs are excellent tools to prove:

- mathematical statements
- correctness of programs/algorithms (that utilize loops and recursive approaches)
- structural results, such as, in trees and other graphs.

Proof by Induction

Show that the statement P(n) is true.

There are basically three steps in an inductive proof:

Step 1: Base case – The base case establishes that the statement is true for the initial/first value.

Step 2: Inductive hypothesis – We assume that the statement is true for some k.

Step 3: The inductive step (proof) – Based on the assumption, we prove that the statement must be true for $k+1$.

Proof by Induction

Walkthrough example: Prove the following using induction.

"The sum 1 + 2 + … + n is equivalent to $\frac{n(n+1)}{2}$."

We need to prove

$$P(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Base Case:

Does our statement hold for the first value of n, which is $n = 1$?

$$P(1) = 1 = \frac{1 \times 2}{2} = 1.$$

Yes, so the base case is satisfied.

Proof by Induction

Inductive hypothesis:

We assume that the statement is true for some k. So, we assume the following is true.

$$P(k) = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Now, using the base case and this assumption, we need to prove that the statement is also true for $k+1$.

Proof by Induction

Proving Inductive hypothesis:

We need to show that it is indeed the case that $P(k+1) = \frac{(k+1)(k+2)}{2}$.

Now,

$$P(k+1) = \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} (i) + (k+1)$$

$\sum_{i=1}^{k} (i) \rightarrow P(k)$

From our assumption, we know that $P(k) = \frac{k(k+1)}{2}$.

So, plugging this above, we get

$$P(k+1) = P(k) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$$

Hence, we prove the desired result.

Proof by Induction

Example: Prove that for $n \geq 4$, $2^n \geq 3n$.

Base Case: $n = 4$

(We see if the theorem statement is true or not for the base case)

$$2^n = 2^4 = 16 \geq 12 = 3 \times 4$$

Thus, for $n = 4$,

$$2^n \geq 3n$$

Inductive step:

We assume that the theorem is true for any $4 \leq n \leq k$, that is,

$$2^k \geq 3k.$$

Proof by Induction

Inductive step (cont.):

Next we need to show that the theorem is true for $n = k+1$. More precisely, we need to show that

If $2^k \geq 3k$, then $2^{(k+1)} \geq 3(k+1)$

So, lets try to prove the above,

$$
\begin{aligned}
2^{(k+1)} &= 2 \times 2^k \\
&\geq 2 \times 3k = 3k + 3k \\
&\geq 3k + 3 \text{ (because } k \geq 4) \\
&= 3(k+1).
\end{aligned}
$$

Hence, we showed that $2^{(k+1)} \geq 3(k+1)$, which proves the desired result.

Proof by Induction – More Examples

Example: Prove that for any positive integer n

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Base Case: $n = 1$

(We see if the theorem statement is true or not for the base case)

$$\text{L.H.S} = \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2}$$

$$\text{R.H.S} = \frac{n}{n+1} = \frac{1}{2}$$

Hence, the statement is true for $n = 1$.

Proof by Induction – More Examples

Inductive step (cont.):

- We assume that the theorem is true for $n = k$, that is,

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

- Next we need to show that the theorem is true for $n = k+1$. More precisely, we need to show that

If $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ is true, then $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$

Proof by Induction – More Examples

Inductive step (cont.):

If $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ is true, then $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \left(\frac{1}{(k+1)(k+2)}\right)$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

Summary of (Weak) Induction

Prove P(n) for $n \geq 1$

Base Case: Show P(1) is true

Inductive Hypothesis:

- Pick some k
- Assume P(i) holds for $i = k$, i.e., P(k) is true for $i = k$.

Inductive Step: Show that P(k) → P(k+1)

(Strong) Induction

Prove P(n) for $n \geq 1$

Base Case: Show P(1) is true

Inductive Hypothesis:

- Pick some k
- Assume P(i) holds for all $i \leq k$. i.e., P(i) is true $\forall\, i \leq k$

Inductive Step: Show that P(k) → P(k+1)

Strong Induction

Difference between weak and strong induction

- The only difference is in the induction hypothesis.
- In weak induction, we only assume that particular statement holds at k^{th} step, while in strong induction, we assume that the particular statement holds at all the steps from the base case to k^{th} step.

Strong Induction – Example

Example: Prove using (strong) induction:

"P(n) = n can be written as the product of prime numbers, where n is an integer ≥ 2."

Base Case: $n = 2$

2 is a prime, so it is a product of a single prime. P(2) holds.

(Strong) Inductive Hypothesis:

Assume that **for all** integers less than or equal to k, the statement holds, that is P(i) holds for all $2 \leq i \leq k$.

Strong Induction – Example

Inductive Step:

Show that P(k+1) is true. More precisely,

$$P(i) \text{ is true (for all } 2 \le i \le k) \quad \rightarrow \quad P(k+1)$$

Case (a): Suppose k+1 is prime, then P(k+1) is true.

Case (b): Suppose k+1 is not a prime, then

$$k+1 = a \times b$$

Clearly, both $2 \le a, b \le k$

(Note that a and b might not necessarily be k.)

Strong Induction – Example

- By the (strong) inductive hypothesis, we know that both a and b can be written as the product of primes.
- Consequently, k+1 can be written as the product of primes.
- Thus, P(k+1) is true, and the statement is true.

Observe that *weak inductive hypothesis* would not have worked here.

Strong Induction – Example

Example:

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

In other words,

P(n) = Postage of n cents can be formed using 4-cent and 5-cent stamps.

Prove that P(n) is true for all $n \geq 12$.

Strong Induction – Example

Base Case: $n = 12$, $n = 13$, $n = 14$, $n = 15$

$12 = 4 + 4 + 4$; $13 = 4 + 4 + 5$

$14 = 4 + 5 + 5$; $15 = 5 + 5 + 5$

Strong Inductive Hypothesis:

Assume P(i) is true for all $12 \le i \le k$.

Inductive Step:

Show that P(k+1) is true.

We can form the postage of (k+1) cents, using the stamps that form postage for (k – 3) cents by adding a 4-cent stamp. QED.

Strong Induction – Example

- Looking back, do you see why we had to prove the base case for four numbers.
- This was necessary because we make the assumption (in the proof in the inductive step) that the statement is true for $(k - 3)$. If we had only proven that P(12) is true, the base case would not be sufficient to support the claim we make in the induction step.