# VANDERBILT UNIVERSITY $\sqrt[5]{3}$ School of Engineering 

## Discrete Structures CS 2212 <br> (Fall 2020)

## 16 - Induction and Recurrence

## Reminders and Recap

## Previously:

Sequences

## Summations

## Today:

## Closed form of Summations

- Substitution method
- Cancellation method


## Induction

## Summations and Closed Forms

## Last time (Summary):

Given a sequence (terms): $a_{1}, a_{2}, a_{3}, \ldots a_{n}$
Find sequence sum:

$$
a_{1}+a_{2}+a_{3}+\ldots+a_{n}
$$



Closed form in terms of $n$
[ Expanded form ]
Find pattern $i^{\text {th }}$ term $\left(a_{i}\right)$
[Summation ]
Simplify
[ Closed form ]

## Summations and Recurrence Relation

## Next:

What if sequence is given in the form of a recurrence relation (instead of individual terms) ?

$$
\begin{aligned}
& a(n)=c_{1} a(n-1)+c_{2}
\end{aligned}
$$

How can we find the closed form of the recurrence relation $a(n)$ ?

## Summations and Closed Forms

## Approach:



Closed form in terms of $n$
[ Recurrence relation]

[ Summation form ]

Simplify
[ Closed form ]

## Recurrence Relations: Substitution Method

Given a recurrence relation:

$$
a(n)=a(n-1)+2 n ; \quad a(0)=1
$$

Find the closed form of $a(n)$.

## Step 1:

Work backwards to unravel and write out some terms.

$$
\begin{array}{ll}
a(n) & =a(n-1)+2 n \\
a(n-1) & =a(n-2)+2(n-1) \\
a(n-2) & =a(n-3)+2(n-2)
\end{array}
$$

## Recurrence Relations: Substitution Method

Step 1: Work backwards to unravel and write out some terms.

$$
\begin{array}{ll}
a(n) & =a(n-1)+2 n \\
a(n-1) & =a(n-2)+2(n-1) \\
a(n-2) & =a(n-3)+2(n-2)
\end{array}
$$

Step 2: Use substitutions and plug in for the terms.

$$
\begin{aligned}
& a(n)=a(n-1)+2 n \\
& a(n)=a(n-2)+2(n-1)+2 n
\end{aligned}
$$

## Recurrence Relations: Substitution Method

Step 1: Work backwards to unravel and write out some terms.

$$
\begin{array}{ll}
a(n) & =a(n-1)+2 n \\
a(n-1) & =a(n-2)+2(n-1) \\
a(n-2) & =a(n-3)+2(n-2)
\end{array}
$$

Step 2: Use substitutions and plug in for the terms.

$$
\begin{aligned}
& a(n)=a(n-1)+2 n \\
& a(n)=a(n-2)+2(n-1)+2 n \\
& a(n)=a(n-3)+2(n-2)+2(n-1)+2 n
\end{aligned}
$$

## Recurrence Relations: Substitution Method

Step 3: Look for a pattern among the terms and visualize the terms over its entirety.

$$
\begin{gathered}
a(n)=a(n-3)+2(n-2)+2(n-1)+2 n \\
i^{\text {th }} \text { term }=\mathbf{2} \boldsymbol{i}
\end{gathered}
$$

Step 4: Establish the summation formula and then determine the closed form

$$
\sum_{i=1}^{n}(2 i)=2\left(\frac{n(n+1)}{2}\right)=n^{2}+n
$$

So, the final closed form would be: $\boldsymbol{n}^{2}+\boldsymbol{n}+\mathbf{1}=\mathrm{O}\left(n^{2}\right)$

## Substitution Method - More Examples

Practice: Analyze how much work a recursive algorithm does by precisely calculating its closed form based on the following recurrence relation.

$$
\mathrm{r}(\mathrm{n})=2 \mathrm{r}(\mathrm{n}-1)+1 ; \mathrm{r}(0)=1
$$

## Recurrence Relations: Substitution Method

Step 1: Work backwards and write out some terms:

$$
\begin{aligned}
& \mathrm{r}(\mathrm{n})=2 \mathrm{r}(\mathrm{n}-1)+1 \\
& \mathrm{r}(\mathrm{n}-1)=2 \mathrm{r}(\mathrm{n}-2)+1 \\
& \mathrm{r}(\mathrm{n}-2)=2 \mathrm{r}(\mathrm{n}-3)+1 \\
& \mathrm{r}(\mathrm{n}-3)=2 \mathrm{r}(\mathrm{n}-4)+1
\end{aligned}
$$

Step 2: Use substitutions for the terms...

$$
\begin{aligned}
\mathrm{r}(\mathrm{n}) & =2 \mathrm{r}(\mathrm{n}-1)+1 \\
& =2[2 \mathrm{r}(\mathrm{n}-2)+1]+1 \\
& =2^{2} \mathrm{r}(\mathrm{n}-2)+2+1
\end{aligned}
$$

Recurrence Relations: Substitution Method

$$
\begin{aligned}
& \text { Step } 2 \text { (con't): } \\
& \begin{aligned}
\mathrm{r}(\mathrm{n}) & =2 \mathrm{r}(\mathrm{n}-1)+1 \\
& =2[2 \mathrm{r}(\mathrm{n}-2)+1]+1
\end{aligned}
\end{aligned}
$$

## Recurrence Relations: Substitution Method

Step 2 (con't):
$\mathrm{r}(\mathrm{n})=2 \mathrm{r}(\mathrm{n}-1)+1$
$=2[2 \mathrm{r}(\mathrm{n}-2)+1]+1$
$=2^{2} \mathrm{r}(\mathrm{n}-2)+2+1$
$=2^{2}[2 \mathrm{r}(\mathrm{n}-3)+1]+2+1$
$=2^{3} r(n-3)+2^{2}+2+1$
Do you see a pattern? This looks like a geometric sequence (there's a summation formula for that).

$$
2^{0}+2^{1}+2^{2}+2^{3}+\ldots+2^{n}
$$

## Recurrence Relations: Substitution Method

Step 4: Try to establish the summation formula and determine the closed form.

$$
\sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{a}^{\mathrm{k}}=\frac{\mathrm{a}^{\mathrm{n}+1}-1}{\mathrm{a}-1} ; \mathrm{a} \neq 1
$$

This leads us to a closed form of $2^{\mathrm{n}+1}-1$ which is $\mathbf{O}\left(\mathbf{2}^{\mathrm{n}}\right)$.

## Substitution Method - More Examples

Practice: Analyze how much work a recursive algorithm does by precisely calculating its closed form based on the following recurrence relation.

$$
g(n)=g(n-1)+2 n-1 ; \quad g(0)=0
$$

## Substitution Method - More Examples

Step 1: Work backwards and write out some terms:

$$
\begin{aligned}
& g(n)=g(n-1)+2 n-1 \\
& g(n-1)=g(n-2)+2(n-1)-1 \\
& g(n-2)=g(n-3)+2(n-2)-1
\end{aligned}
$$

Step 2: Use substitutions for the terms...

$$
\begin{aligned}
& g(n)=g(n-1)+2 n-1 \\
& g(n)=[g(n-2)+2(n-1)-1]+2 n-1
\end{aligned}
$$

## Substitution Method - More Examples

Step 1: Work backwards and write out some terms:

$$
\begin{aligned}
& g(n)=g(n-1)+2 n-1 \\
& g(n-1)=g(n-2)+2(n-1)-1 \\
& g(n-2)=g(n-3)+2(n-2)-1
\end{aligned}
$$

Step 2: Use substitutions for the terms...

$$
\begin{aligned}
& g(n)=g(n-1)+2 n-1 \\
& g(n)=[g(n-2)+2(n-1)-1]+2 n-1 \\
& g(n)=[g(n-3)+2(n-2)-1]+2(n-1)-1+2 n-1
\end{aligned}
$$

## Substitution Method - More Examples

Step 3: Look for a pattern among the terms and visualize the terms over its entirety.
(Tip: It often helps to reorganize and simplify a bit first.)

$$
g(n)=g(n-3)+2(n-2)-1+2(n-1)-1+2 n-1
$$

Group the 1's together.

$$
g(n)=g(n-3)+2(n-2)+2(n-1)+2 n-\mathbf{3}
$$

Let's now visualize the above pattern over all the terms in the recursion and see what that looks like

## Substitution Method - More Examples

(Tip: Sometimes it helps to think backwards.)
$g(n)$
$=2(n-(n-1))+2(n-(n-2))+\ldots+2(n-2)+2(n-1)+2 n-n$
$=2(1)+2(2)+\ldots+2(n-2)+2(n-1)+2 n-n$
Step 4: Find the summation formula and then closed form
Do you see the summation pattern? It looks like this...

$$
\sum_{k=1}^{n}(2 k)-n
$$

## Substitution Method - More Examples

Use your summation rules to help derive the closed form.

$$
\begin{aligned}
& \sum_{k=1}^{n}(2 k)-n \\
& =2 \sum_{k=1}^{n}(k)-n \\
& =2\left(\frac{n(n+1)}{2}\right)-n=n^{2}+n-n=n^{2}
\end{aligned}
$$

We conclude this is an $\mathbf{O}\left(\mathbf{n}^{2}\right)$ recursive algorithm.

## Recurrence Relations: Cancellation Method

Another technique that is useful for solving a recurrence relation is the:

## Cancellation method.

## Summations and Closed Forms

## Approach:



Closed form in terms of $n$
[ Recurrence relation]

[ Summation form ]

Simplify
[ Closed form ]

## Summations and Closed Forms

## Approach:



Closed form in terms of $n$
[ Recurrence relation]

[ Summation form ]

Simplify
[ Closed form ]

## Recurrence Relations: Cancellation Method

How it works:

1. Start with the general (recurrence) equation $\mathbf{r}(\mathbf{n})$ and continue writing new equations whose left side is the term involving the recurrence $\mathbf{r}$ on the right side of the previous equation.
2. Look for a pattern that allows us to skip ahead and write the last equation $\mathbf{r}(\mathbf{0})$ on the right side.
3. Add up the equations and cancel the like terms to obtain an expression for $\mathbf{r}(\mathbf{n})$.

## Recurrence Relations: Cancellation Method

Example: Solve the following recurrence using cancellation:

$$
\begin{aligned}
& a(0)=1 \\
& a(n)=a(n-1)+2 n
\end{aligned}
$$

Step 1: As with substitution, we start by writing out some terms:

$$
\begin{aligned}
& a(n)=a(n-1)+2 n \\
& a(n-1)=a(n-2)+2(n-1) \\
& a(n-2)=a(n-3)+2(n-2) \\
& \cdots \\
& a(n-(n-1))=a(0)+2(n-(n-1)) \text { same as } a(1)=a(0)+2
\end{aligned}
$$

## Recurrence Relations: Cancellation Method

Step 2: Next we add up the terms on both sides of the = sign

$$
\begin{aligned}
& a(n)+a(n 1)+a(n-2)+a(n-3)+\ldots+a(1)+a(0) \\
& = \\
& a(n-1)+2 n+a(n-2)+2(n-1)+a(n-3)+2(n-2) \\
& +\ldots+a(0)+2+a(0)
\end{aligned}
$$

Step 3: Cancel the like terms to obtain the equation for $\mathbf{a}(\mathbf{n})$.

$$
a(n)=a(0)+2(1)+\ldots+2(n-2)+2(n-1)+2 n
$$

## Recurrence Relations: Cancellation Method

Step 4: Determine the summation and closed form.

$$
\begin{gathered}
a(n)=a(0)+2(1)+\ldots+2(n-2)+2(n-1)+2 n \\
a(n)=1+2[1+\ldots+(n-2)+(n-1)+n] \\
a(n)=1+2\left[\frac{n(n+1)}{2}\right] \\
a(n)=n^{2}+n+1
\end{gathered}
$$

This is an $O\left(\mathrm{n}^{2}\right)$ algorithm

Mathematical Induction

## Mathematical Induction

## General Idea of Induction:

Show that a person can provably climb to any arbitrary step.
For this, all we need to show is

1) A person can climb the first step.
2) A person knows to climb from any step $k$ to the next step $k+1$.


Part 1


Part 2

## Induction - A powerful proof technique

An inductive proof establishes that some statement $\mathrm{P}(n)$ parameterized by $n$ is true, for any positive $n$.

Inductive proofs are excellent tools to prove:

- mathematical statements
- correctness of programs/algorithms (that utilize loops and recursive approaches)
- structural results, such as, in trees and other graphs.


## Proof by Induction

## Show that the statement $P(n)$ is true.

There are basically three steps in an inductive proof:
Step 1: Base case - The base case establishes that the statement is true for the initial/first value.

Step 2: Inductive hypothesis - We assume that the statement is true for some $k$.

Step 3: The inductive step (proof) - Based on the assumption, we prove that the statement must be true for $k+1$.

## Proof by Induction

Walkthrough example: Prove the following using induction.
"The sum $1+2+\ldots+n$ is equivalent to $\frac{n(n+1)}{2}$."
We need to prove

$$
\mathrm{P}(n)=\sum_{i=1}^{n} i=\frac{n(n+1)}{2}
$$

## Base Case:

Does our statement hold for the first value of $n$, which is $n=1$ ?

$$
P(1)=1=\frac{1 \times 2}{2}=1
$$

Yes, so the base case is satisfied.

## Proof by Induction

## Inductive hypothesis:

We assume that the statement is true for some $k$. So, we assume the following is true.

$$
\mathrm{P}(k)=\sum_{i=1}^{k} i=\frac{k(k+1)}{2}
$$

Now, using the base case and this assumption, we need to prove that the statement is also true for $k+1$.

## Proof by Induction

## Proving Inductive hypothesis:

We need to show that it is indeed the case that $\mathrm{P}(k+1)=\frac{(k+1)(k+2)}{2}$.
Now,

$$
\mathrm{P}(k+1)=\sum_{i=1}^{k+1} i=\sum_{i=1}^{k}(i)+(k+1)
$$

From our assumption, we know that $\mathrm{P}(k)=\frac{k(k+1)}{2}$.
So, plugging this above, we get

$$
\mathrm{P}(k+1)=\mathrm{P}(k)+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{k^{2}+3 k+2}{2}=\frac{(k+1)(k+2)}{2}
$$

Hence, we prove the desired result.

## Proof by Induction

Example: Prove that for $n \geq 4,2^{n} \geq 3 n$.
Base Case: $n=4$
(We see if the theorem statement is true or not for the base case)

$$
2^{n}=2^{4}=16 \geq 12=3 \times 4
$$

Thus, for $n=4$,

$$
2^{n} \geq 3 n
$$

## Inductive step:

We assume that the theorem is true for any $4 \leq n \leq k$, that is,

$$
2^{k} \geq 3 k
$$

## Proof by Induction

## Inductive step (cont.):

Next we need to show that the theorem is true for $n=k+1$. More precisely, we need to show that

$$
\text { If } 2^{k} \geq 3 k \text {, then } 2^{(k+1)} \geq 3(k+1)
$$

So, lets try to prove the above,

$$
\begin{aligned}
2^{(k+1)} & =2 \times 2^{k} \\
& \geq 2 \times 3 k=3 k+3 k \\
& \geq 3 k+3 \text { (because } k \geq 4) \\
& =3(k+1) .
\end{aligned}
$$

Hence, we showed that $2^{(k+1)} \geq 3(k+1)$, which proves the desired result.

## Proof by Induction - More Examples

Example: Prove that for any positive integer n

$$
\sum_{i=1}^{n} \frac{1}{i(i+1)}=\frac{n}{n+1}
$$

Base Case: $n=1$
(We see if the theorem statement is true or not for the base case)

$$
\begin{aligned}
& \text { L.H.S }=\sum_{i=1}^{1} \frac{1}{i(i+1)}=\frac{1}{2} \\
& \text { R.H.S }=\frac{n}{n+1}=\frac{1}{2}
\end{aligned}
$$

Hence, the statement is true for $n=1$.

## Proof by Induction - More Examples

## Inductive step (cont.):

- We assume that the theorem is true for $n=k$, that is,

$$
\sum_{i=1}^{k} \frac{1}{i(i+1)}=\frac{k}{k+1}
$$

- Next we need to show that the theorem is true for $n=k+1$. More precisely, we need to show that

$$
\text { If } \sum_{i=1}^{k} \frac{1}{i(i+1)}=\frac{k}{k+1} \text { is true, then } \sum_{i=1}^{k+1} \frac{1}{i(i+1)}=\frac{k+1}{k+2}
$$

## Proof by Induction - More Examples

Inductive step (cont.):

$$
\text { If } \sum_{i=1}^{k} \frac{1}{i(i+1)}=\frac{k}{k+1} \text { is true, then } \sum_{i=1}^{k+1} \frac{1}{i(i+1)}=\frac{k+1}{k+2}
$$

$$
\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} & =\sum_{i=1}^{k} \frac{1}{i(i+1)}+\left(\frac{1}{(k+1)(k+2)}\right) \\
& =\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \\
& =\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^{2}+2 k+1}{(k+1)(k+2)}=\frac{(k+1)^{2}}{(k+1)(k+2)} \\
& =\frac{k+1}{k+2}
\end{aligned}
$$

## Summary of (Weak) Induction

Prove $P(n)$ for $n \geq 1$

Base Case: Show $\mathrm{P}(1)$ is true

## Inductive Hypothesis:

- Pick some k
- Assume $\mathrm{P}(i)$ holds for $i=k$, i.e., $\mathrm{P}(k)$ is true for $i=k$.

Inductive Step: Show that $\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)$

## (Strong) Induction

Prove $P(n)$ for $n \geq 1$

Base Case: Show $\mathrm{P}(1)$ is true

## Inductive Hypothesis:

- Pick some $k$
- Assume $\mathrm{P}(i)$ holds for all $i \leq k$. i.e., $\mathrm{P}(i)$ is true $\forall i \leq k$

Inductive Step: Show that $\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)$

## Strong Induction

## Difference between weak and strong induction

- The only difference is in the induction hypothesis.
- In weak induction, we only assume that particular statement holds at $k^{\text {th }}$ step, while in strong induction, we assume that the particular statement holds at all the steps from the base case to $k^{\text {th }}$ step.


## Strong Induction - Example

Example: Prove using (strong) induction:
" $\mathrm{P}(n)=n$ can be written as the product of prime numbers, where $n$ is an integer $\geq 2$."

## Base Case: $n=2$

2 is a prime, so it is a product of a single prime. $\mathrm{P}(2)$ holds.

## (Strong) Inductive Hypothesis:

Assume that for all integers less than or equal to $k$, the statement holds, that is $\mathrm{P}(i)$ holds for all $2 \leq i \leq k$.

## Strong Induction - Example

## Inductive Step:

Show that $\mathrm{P}(k+1)$ is true. More precisely,

$$
\mathrm{P}(i) \text { is true }(\text { for all } 2 \leq i \leq k) \rightarrow \mathrm{P}(k+1)
$$

Case (a): Suppose $k+1$ is prime, then $\mathrm{P}(k+1)$ is true.
Case (b): Suppose $k+1$ is not a prime, then

$$
k+1=a \times b
$$

Clearly, both $2 \leq a, b \leq k$
(Note that $a$ and $b$ might not necessarily be $k$.)

## Strong Induction - Example

- By the (strong) inductive hypothesis, we know that both $a$ and $b$ can be written as the product of primes.
- Consequently, $k+1$ can be written as the product of primes.
- Thus, $\mathrm{P}(k+1)$ is true, and the statement is true.

Observe that weak inductive hypothesis would not have worked here.

## Strong Induction - Example

## Example:

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

In other words,
$\mathrm{P}(n)=$ Postage of $n$ cents can be formed using 4 -cent and 5 -cent stamps.

Prove that $P(n)$ is true for all $\boldsymbol{n} \geq 12$.


## Strong Induction - Example

Base Case: $n=12, n=13, n=14, n=15$
$12=4+4+4 ;$
$13=4+4+5$
$14=4+5+5 ;$
$15=5+5+5$


## Strong Inductive Hypothesis:

Assume $\mathrm{P}(i)$ is true for all $12 \leq i \leq k$.

## Inductive Step:

Show that $\mathrm{P}(k+1)$ is true.
We can form the postage of $(k+1)$ cents, using the stamps that form postage for $(k-3)$ cents by adding a 4 -cent stamp. QED.

## Strong Induction - Example

- Looking back, do you see why we had to prove the base case for four numbers.
- This was necessary because we make the assumption (in the proof in the inductive step) that the statement is true for $(k-3)$. If we had only proven that $\mathrm{P}(12)$ is true, the base case would not be sufficient to support the claim we make in the induction step.

