



School of Engineering

#### Discrete Structures CS 2212 (Fall 2020)

**16 – Induction and Recurrence** 

### **Reminders and Recap**

### Previously: Sequences Summations

#### **Today:**

#### **Closed form of Summations**

- Substitution method
- Cancellation method

#### Induction

#### **Summations and Closed Forms**

#### Last time (Summary):





### **Summations and Recurrence Relation**

Next:

# What if sequence is given in the form of a **recurrence relation** (instead of individual terms) ?

$$a(n) = c_1 a(n-1) + c_2$$



How can we find the closed form of the recurrence relation a(n)?

### **Summations and Closed Forms**

**Approach:** 



Given a recurrence relation:

a(n) = a(n-1) + 2n; a(0) = 1

Find the closed form of a(n).

#### **Step 1:**

Work **backwards** to *unravel* and write out some terms.

$$a(n) = a(n-1) + 2n$$
  

$$a(n-1) = a(n-2) + 2(n-1)$$
  

$$a(n-2) = a(n-3) + 2(n-2)$$

Work **backwards** to *unravel* and write out some terms.

a(n) = a(n-1) + 2n a(n-1) = a(n-2) + 2(n-1)a(n-2) = a(n-3) + 2(n-2)

**Step 1:** 

**Step 2:** 

Use **substitutions** and plug in for the terms.

$$a(n) = a(n-1) + 2n$$
  
 $a(n) = a(n-2) + 2(n-1) + 2n$ 

Work **backwards** to *unravel* and write out some terms.

a(n) = a(n-1) + 2n a(n-1) = a(n-2) + 2(n-1)a(n-2) = a(n-3) + 2(n-2)

**Step 1:** 

**Step 2:** 

Use **substitutions** and plug in for the terms.

$$a(n) = a(n-1) + 2n$$
  

$$a(n) = a(n-2) + 2(n-1) + 2n$$
  

$$a(n) = a(n-3) + 2(n-2) + 2(n-1) + 2n$$

**<u>Step 3:</u>** Look for a **pattern** among the terms and visualize the terms over its entirety.

$$a(n) = a(n-3) + 2(n-2) + 2(n-1) + 2n$$

$$i^{th}$$
 term = **2** i

**Step 4:** Establish the summation formula and then determine the closed form  $\sum_{i=1}^{n} (2i) = 2\left(\frac{n(n+1)}{2}\right) = n^2 + n$ So, the final closed form would be:  $n^2 + n + 1 = O(n^2)$ Add 1 for the initial term a(0)

**Practice:** Analyze how much work a recursive algorithm does by precisely calculating its closed form based on the following recurrence relation.

r(n) = 2r(n-1) + 1; r(0) = 1

**Step 1:** Work backwards and write out some terms:

r(n) = 2r(n-1) + 1 r(n-1) = 2r(n-2) + 1 r(n-2) = 2r(n-3) + 1r(n-3) = 2r(n-4) + 1

**Step 2:** Use substitutions for the terms...

$$r(n) = 2r(n-1) + 1$$
  
= 2[2r(n-2) + 1] + 1

$$= 2^2 r(n-2) + 2 + 1$$

Step 2 (con't): r(n) = 2r(n-1) + 1= 2[2r(n-2) + 1] + 1

### Step 2 (con't):

- r(n) = 2 r(n-1) + 1
  - = 2 [2r(n-2) + 1] + 1

$$= 2^2 r(n-2) + 2 + 1$$

$$= 2^{2} \left[ \left[ 2r(n-3) + 1 \right] \right] + 2 + 1$$

 $= 2^3 r(n-3) + 2^2 + 2 + 1$ 

Do you see a pattern? This looks like a geometric sequence (there's a summation formula for that).

 $2^0 + 2^1 + 2^2 + 2^3 + \dots + 2^n$ 

**Step 4:** Try to establish the summation formula and determine the closed form.

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}; a \neq 1$$

This leads us to a closed form of  $2^{n+1} - 1$  which is  $O(2^n)$ .

**Practice:** Analyze how much work a recursive algorithm does by precisely calculating its closed form based on the following recurrence relation.

g(n) = g(n-1) + 2n - 1; g(0) = 0

**Step 1:** Work backwards and write out some terms:

g(n)	=	g(n-1) + 2n - 1	
g(n-1)	=	g(n-2) + 2(n-1) -	

$$g(n-2) = g(n-3) + 2(n-2) - 1$$

**Step 2:** Use substitutions for the terms...

$$g(n) = g(n-1) + 2n - 1$$
  

$$g(n) = [g(n-2) + 2(n-1) - 1] + 2n - 1$$

1

**Step 1:** Work backwards and write out some terms:

g(n)	=	g(n-1) + 2n - 1
g(n-1)	=	g(n-2) + 2(n-1) - 2

$$g(n-2) = g(n-3) + 2(n-2) - 1$$

**Step 2:** Use substitutions for the terms...

$$g(n) = g(n-1) + 2n - 1$$

$$g(n) = [g(n-2) + 2(n-1) - 1] + 2n - 1$$

$$g(n) = [[g(n-3) + 2(n-2) - 1]] + 2(n-1) - 1 + 2n - 1$$

**Step 3:** Look for a pattern among the terms and visualize the terms over its entirety.

(**Tip:** *It often helps to reorganize and simplify a bit first.*)

g(n) = g(n-3) + 2(n-2) - 1 + 2(n-1) - 1 + 2n - 1

Group the **1**'s together.

g(n) = g(n-3) + 2(n-2) + 2(n-1) + 2n - 3

Let's now visualize the above pattern over all the terms in the recursion and see what that looks like

(Tip: Sometimes it helps to think backwards.)

g(n)

 $= 2(n-(n-1)) + 2(n-(n-2)) + \dots + 2(n-2) + 2(n-1) + 2n - n$  $= 2(1) + 2(2) + \dots + 2(n-2) + 2(n-1) + 2n - n$ 

**Step 4:** Find the summation formula and then closed form

Do you see the summation pattern? It looks like this...

$$\sum_{k=1}^{n} (2k) - n$$

Use your summation rules to help derive the closed form.

$$\sum_{k=1}^{n} (2k) - n$$

$$= 2 \sum_{k=1}^{n} (k) - n$$

$$= 2(\frac{n(n+1)}{2}) - n = n^2 + n - n = n^2$$

We conclude this is an  $O(n^2)$  recursive algorithm.

# Another technique that is useful for solving a recurrence relation is the:

## **Cancellation method**.

### **Summations and Closed Forms**

**Approach:** 



#### **Summations and Closed Forms**

**Approach:** 



#### How it works:

• Same as the substitution method

- Start with the general (recurrence) equation r(n) and continue writing new equations whose left side is the term involving the recurrence r on the right side of the previous equation.
- 2. Look for a pattern that allows us to skip ahead and write the last equation r(0) on the right side.
- **3.** Add up the equations and **cancel** the like terms to obtain an expression for **r(n)**.

Example: Solve the following recurrence using cancellation: a(0) = 1 a(n) = a(n - 1) + 2n

Step 1: As with substitution, we start by writing out some terms: a(n) = a(n-1) + 2n a(n-1) = a(n-2) + 2(n-1) a(n-2) = a(n-3) + 2(n-2)... a(n-(n-1)) = a(0) + 2(n-(n-1)) same as a(1) = a(0) + 2

**Step 2:** Next we add up the terms on both sides of the = sign

$$a(n) + \frac{a(n-1)}{a(n-1)} + \frac{a(n-2)}{a(n-3)} + \frac{a(n-3)}{a(n-1)} + \frac{a(n-1)}{a(n-2)} + \frac{a(n-2)}{a(n-1)} + \frac{a(n-3)}{a(n-2)} + 2(n-2) + \dots + a(0) + 2 + \frac{a(0)}{a(0)}$$

**Step 3:** Cancel the like terms to obtain the equation for **a(n)**.

$$a(n) = a(0) + 2(1) + ... + 2(n-2) + 2(n-1) + 2n$$

**Step 4:** Determine the **summation** and closed form.

$$a(n) = a(0) + 2(1) + \dots + 2(n-2) + 2(n-1) + 2n$$

$$a(n) = 1 + 2 [1 + \dots + (n-2) + (n-1) + n]$$

$$a(n) = 1 + 2 [\frac{n(n+1)}{2}]$$

$$a(n) = n^{2} + n + 1$$

This is an  $O(n^2)$  algorithm

## Mathematical Induction

### **Mathematical Induction**

#### **General Idea of Induction:**

Show that a person can provably climb to any arbitrary step.

For this, all we need to show is

1) A person can climb the first step.

2) A person knows to climb from any step k to the next step k+1.



### Induction – A powerful proof technique

An inductive proof establishes that some statement P(n) parameterized by n is true, for any positive n.

Inductive proofs are excellent tools to prove:

- mathematical statements
- correctness of programs/algorithms (that utilize loops and recursive approaches)
- **structural results**, such as, in trees and other graphs.

#### Show that the statement P(n) is true.

There are basically three steps in an inductive proof:

**Step 1: Base case** – The base case establishes that the statement is true for the initial/first value.

**Step 2: Inductive hypothesis** – We assume that the statement is true for some *k*.

**Step 3: The inductive step (proof)** – Based on the assumption, we prove that the statement must be true for k+1.

Walkthrough example: Prove the following using induction. "The sum 1 + 2 + ... + n is equivalent to  $\frac{n(n+1)}{2}$ ."

We need to prove

$$P(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

#### **Base Case:**

Does our statement hold for the first value of n, which is n = 1?

$$P(1) = 1 = \frac{1 \times 2}{2} = 1.$$

Yes, so the base case is satisfied.

#### **Inductive hypothesis:**

We assume that the statement is true for some k. So, we assume the following is true.

$$P(k) = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Now, using the base case and this assumption, we need to prove that the statement is also true for k+1.

#### **Proving Inductive hypothesis:**

We need to show that it is indeed the case that  $P(k+1) = \frac{(k+1)(k+2)}{2}$ .

Now,  

$$P(k+1) = \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} (i) + (k+1)$$

$$(k+1) \to P(k)$$

From our assumption, we know that  $P(k) = \frac{k(k+1)}{2}$ .

So, plugging this above, we get  $P(k+1) = P(k) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2}$ Hence, we prove the desired result.

```
Example: Prove that for n \ge 4, 2^n \ge 3n.
```

**Base Case:** n = 4(We see if the theorem statement is true or not for the base case)  $2^n = 2^4 = 16 \ge 12 = 3 \times 4$ Thus, for n = 4,  $2^n \ge 3n$ 

#### **Inductive step:** We assume that the theorem is true for any $4 \le n \le k$ , that is, $2^k \ge 3k$ .

#### Inductive step (cont.):

Next we need to show that the theorem is true for n = k+1. More precisely, we need to show that

If  $2^k \ge 3k$ , then  $2^{(k+1)} \ge 3(k+1)$ 

So, lets try to prove the above,

```
2^{(k+1)} = 2 \times 2^{k}

\geq 2 \times 3k = 3k + 3k

\geq 3k + 3 \text{ (because } k \geq 4\text{)}

= 3(k+1).
```

Hence, we showed that  $2^{(k+1)} \ge 3(k+1)$ , which proves the desired result.

### **Proof by Induction – More Examples**



#### **Base Case:** n = 1

(We see if the theorem statement is true or not for the base case)

L.H.S = 
$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2}$$
  
R.H.S =  $\frac{n}{n+1} = \frac{1}{2}$   
Hence, the statement is true for  $n = 1$ .

### **Proof by Induction – More Examples**

#### Inductive step (cont.):

• We assume that the theorem is true for n = k, that is,

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Next we need to show that the theorem is true for n = k+1.
 More precisely, we need to show that

If 
$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$
 is true, then  $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$ 

### **Proof by Induction – More Examples**

#### Inductive step (cont.):

If 
$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$
 is true, then  $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$ 

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \left(\frac{1}{(k+1)(k+2)}\right)$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2}$$

### **Summary of (Weak) Induction**

```
Prove P(n) for n \ge 1
```

#### **Base Case:** Show P(1) is true

#### **Inductive Hypothesis:**

- Pick some k
- Assume P(i) holds for i = k, i.e., P(k) is true for i = k.

**Inductive Step:** Show that  $P(k) \rightarrow P(k+1)$ 

### (Strong) Induction

Prove P(n) for  $n \ge 1$ 

#### **Base Case:** Show P(1) is true

#### **Inductive Hypothesis:**

- Pick some k
- Assume P(i) holds for all  $i \le k$ . i.e., P(i) is true  $\forall i \le k$

**Inductive Step:** Show that  $P(k) \rightarrow P(k+1)$ 

### **Strong Induction**

#### **Difference between weak and strong induction**

- The only difference is in the induction hypothesis.
- In weak induction, we only assume that particular statement holds at  $k^{\text{th}}$  step, while in strong induction, we assume that the particular statement holds at all the steps from the base case to  $k^{\text{th}}$  step.

**Example:** Prove using <u>(strong) induction:</u> "P(n) = n can be written as the product of prime numbers, where n is an integer  $\ge 2$ ."

#### **Base Case:** *n* = 2

2 is a prime, so it is a product of a single prime. P(2) holds.

#### (Strong) Inductive Hypothesis:

Assume that **for all** integers less than or equal to k, the statement holds, that is P(i) holds for all  $2 \le i \le k$ .

#### **Inductive Step:**

Show that P(k+1) is true. More precisely, P(i) is true (for all  $2 \le i \le k$ )  $\rightarrow$  P(k+1)

**Case (a):** Suppose k+1 is prime, then P(k+1) is true.

**Case (b):** Suppose k+1 is not a prime, then

 $k+1 = a \times b$ 

Clearly, both  $2 \le a, b \le k$ 

(Note that a and b might not necessarily be k.)

- By the (strong) inductive hypothesis, we know that both *a* and *b* can be written as the product of primes.
- Consequently, *k*+1 can be written as the product of primes.
- Thus, P(k+1) is true, and the statement is true.

Observe that *weak inductive hypothesis* would not have worked here.

#### **Example:**

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

In other words,

P(n) = Postage of *n* cents can be formed using 4-cent and 5-cent stamps.

Prove that P(n) is true for all  $n \ge 12$ .



Base Case:
 
$$n = 12$$
,  $n = 13$ ,  $n = 14$ ,  $n = 15$ 
 $12 = 4 + 4 + 4$ ;
  $13 = 4 + 4 + 5$ 
 $14 = 4 + 5 + 5$ ;
  $15 = 5 + 5 + 5$ 



#### **Strong Inductive Hypothesis:**

Assume P(*i*) is true for all  $12 \le i \le k$ .

#### **Inductive Step:**

Show that P(k+1) is true.

We can form the postage of (k+1) cents, using the stamps that form postage for (k-3) cents by adding a 4-cent stamp. QED.

- Looking back, do you see why we had to prove the base case for four numbers.
- This was necessary because we make the assumption (in the proof in the inductive step) that the statement is true for (k-3). If we had only proven that P(12) is true, the base case would not be sufficient to support the claim we make in the induction step.