Math 3331 ODEs - ST3 Solns

\# *1* A spring is stretched 20 cm by a 4-kg mass. The weight is pulled down an additional 1 m and released with an upward velocity of 4 m/s. Find the position of the mass at and time t.

Soln. Since $F = ma = kx$ then we have $4 \times 9.8 = k \times 0.2$ giving $k = 196$. The equation of motion is given by $m\ddot{x} + kx = 0$ or in this case $4\ddot{x} + 196x = 0$ which simplifies to $\ddot{x} + 49x = 0$. The solution is $x = c_1 \cos 7t + c_2 \sin 7t$. The initial conditions are: $x(0) = 1$ and $\dot{x}(0) = -4$. Imposing these gives $x(0) = c_1 = 1$ and $\dot{x}(0) = 7c_2 = -4$ so $c_2 = -4/7$ giving the final solution as

$$x = \cos 7t - \frac{4}{7}\sin 7t. \tag{1}$$

\# *2* A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t.

Soln. Given in the problem is $m = 2$ and $l = 14$. Since $F = 6 = k \times 0.5$ then $k = 12$. The differential equation for the motion is $2\ddot{x} + 14\dot{x} + 12x = 0$ or $\ddot{x} + 7\dot{x} + 6x = 0$. The solution is given by $x = c_1 e^{-t} + c_2 e^{-6t}$. The initial conditions are: $x(0) = 1, \quad \dot{x}(0) = 0$. These give $c_1 + c_2 = 1$ and $-c_1 - 6c_2 = 0$ from which we obtain $c_1 = 6/5, \quad c_2 = -1/5$ giving the final solution as

$$x = \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t}. \tag{2}$$

\# *3* This involves solving 6 systems.

\# *3*(i)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \bar{x}. \tag{3}$$

Here $Tr A = 3$ and $Det A = -4$ and the characteristic equation is

$$\lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 4$.

Case 1: $\lambda = -1$

From $(A - \lambda I)\vec{u} = \vec{0}$ we have

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which we obtain upon expanding $2u_1 + 2u_2 = 0$ and obtain the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and thus, the first solutions is

$$\bar{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Case 2: $\lambda = 4$

From $(A - \lambda I)\vec{u} = \vec{0}$ we have

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which we obtain upon expanding $-3u_1 + 2u_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and the second solution is

$$\bar{x}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}.$$

The general solution to (3) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t}.$$

3(ii)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \bar{x}, \quad \vec{x}(0) = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \tag{4}$$

Here $TrA = 1$ and $DetA = -2$ and the characteristic equation is

$$\lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0,$$

from which we obtain the eigenvalues $\lambda = -1$ and $\lambda = 2$.

Case 1: $\lambda = -1$

From $(A - \lambda I)\vec{u} = \vec{0}$ we have

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which we obtain upon expanding $4u_1 - 2u_2 = 0$ and obtain the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and thus, the first solutions is

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$$

Case 2: $\lambda = 2$

From $(A - \lambda I)\vec{u} = \vec{0}$ we have

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which we obtain upon expanding $u_1 - 2u_2 = 0$ and we deduce the eigenvector

$$\bar{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and the second solution is

$$\bar{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

The general solution to (4) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

From the initial condition we have

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$$

from which we deduce that $c_1 = 2$ and $c_2 = 3$ giving the solution subject to the initial condition as

$$\bar{x} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

3(iii)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \bar{x} \tag{5}$$

In this example, $TrA = 4$ and $DetA = 4$ and the characteristic equation is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0,$$

from which we obtain the repeated eigenvalues $\lambda = 2$. From $(A - \lambda I)\vec{u} = \vec{0}$ we have

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $u_1 + u_2 = 0$ and obtain the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and thus, the first solution is

$$\bar{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

The second solution is given by

$$\bar{x}_1 = \bar{u}\, t\, e^{2t} + \bar{v}\, e^{2t}$$

where $\bar{v}$ satisfies

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

giving $-v_1 - v_2 = 1$. Any choice will suffice so here we will choose

$$\bar{v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

and the second solution is

$$\bar{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t}.$$

The general solution to (5) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{2t}.$$

3(iv)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \bar{x}, \quad \vec{x}(0) = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{6}$$

In this example, $TrA = 6$ and $DetA = 9$ and the characteristic equation is

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0,$$

from which we obtain the repeated eigenvalues $\lambda = 3$. From $(A - \lambda I)\vec{u} = \vec{0}$ we have

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $u_1 - 2u_2 = 0$ and obtain the eigenvector

$$\bar{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and thus, the first solution is

$$\bar{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t}.$$

The second solution is given by

$$\bar{x}_1 = \bar{u}\, t\, e^{3t} + \bar{v}\, e^{3t}$$

where $\bar{v}$ satisfies

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

giving $v_1 - 2v_2 = 1$. Any choice will suffice so here we will choose

$$\bar{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the second solution is

$$\bar{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\, e^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}.$$

The general solution to (6) is then given by

$$\bar{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{2t}.$$

Imposing the initial condition gives

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

giving two equations for c_1 and c_2. This leads to $c_1 = 1$ and $c_2 = -5$ and thus, our solution is

$$\bar{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} - 5 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{2t}.$$

3(v)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \bar{x}. \tag{7}$$

Here, $TrA = 10$ and $DetA = 29$ and the characteristic equation is

$$\lambda^2 - 10\lambda + 29 = 0,$$

from which we obtain the complex eigenvalues $\lambda = 5 \pm 2i$ so $\alpha = 5$ and $\beta = 2$. Choosing the positive case, from $(A - \lambda I)\vec{u} = \vec{0}$ we have

$$\begin{pmatrix} 1 - 2i & -1 \\ 5 & -1 - 2i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $5u_1 - (1+2i)u_2 = 0$ and obtain the eigenvector

$$\bar{u} = \begin{pmatrix} 1+2i \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i$$

So

$$R = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

and our two solutions are

$$\begin{aligned} \bar{x}_1 &= \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] e^{5t} \\ \bar{x}_2 &= \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] e^{5t} \end{aligned}$$

giving the general solution to (7) as

$$\bar{x} = c_1 \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t \right] e^{5t} + c_2 \left[\begin{pmatrix} 1 \\ 5 \end{pmatrix} \sin 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t \right] e^{5t}$$

3(iv)

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 7 & -5 \\ 10 & -3 \end{pmatrix} \bar{x}, \quad \vec{x}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{8}$$

In this example, $TrA = 4$ and $DetA = 29$ and the characteristic equation is

$$\lambda^2 - 4\lambda + 29 = 0,$$

from which we obtain the complex eigenvalues $\lambda = 2 \pm 5i$ so $\alpha = 2$ and $\beta = 5$. Choosing the positive case, from $(A - \lambda I)\vec{u} = \vec{0}$ we have

$$\begin{pmatrix} 5-5i & -5 \\ 10 & -5-5i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we obtain upon expanding $5(1-i)u_1 - 5u_2 = 0$ and obtain the eigenvector

$$\bar{u} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i$$

So

$$R = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and our solutions is

$$\bar{x} = c_1 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 5t \right] e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 5t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 5t \right] e^{2t}.$$

Imposing the initial condition gives

$$\bar{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

which readily gives $c_1 = 3$ and $c_2 = 5$ and thus our solution

$$\bar{x} = 3\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 5t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin 5t\right] e^{2t} + 5\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 5t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 5t\right] e^{2t}.$$