Math 1496 - Sample Test 2

1. Find the derivative of the following

$$\begin{array}{cc} (i) & y = x\sin^{-1} x \\ (ii) & y = \ln(x^2 + e^x) \\ (iii) & x^2 - xy + y^4 = x - y \end{array}$$

Soln (i). $y' = 1\sin^{-1} x + x \cdot \dfrac{1}{\sqrt{1-x^2}}$

Soln (ii). $y' = \dfrac{1}{x^2+e^x} \cdot (2x + e^x)$

Soln (iii). We differentiate implicitly giving

$$\begin{aligned} 2x - (xy' + y) + 4y^3y' &= 1 - y' \\ -xy' + 4y^3y' + y' &= 1 - 2x + y \\ (-x + 4y^3 + 1)y' &= 1 - 2x + y \\ y' &= \frac{1 - 2x + y}{-x + 4y^3 + 1} \end{aligned} \tag{1}$$

$$y' = \frac{1}{x^2+e^x} \cdot (2x + e^x)$$

2. Find the absolute minimum and maximum of the following on the given interval

$$\begin{array}{cc} (i) & f(x) = 1 - x^2 \quad \text{on} \quad [-2, 3] \\ (ii) & f(x) = 2x^3 - 15x^2 + 24x \quad \text{on} \quad [0, 3] \end{array}$$

Soln (i). Since f is continuous on $[-2, 3]$ and differentiable on $(-2, 3)$ it will have a minimum and maximum. These will be located at the endpoints or inside the interval at the critical points. So here $f' = -2x$ and $f' = 0$ when $x = 0$.

$$f(-2) = -3, \quad f(0) = 1(max), \quad f(3) = -8(min). \tag{2}$$

Soln (ii). Since f is continuous on $[0, 3]$ and differentiable on $(0, 3)$ it will have a minimum and maximum. These will be located at the endpoints or inside the interval at the critical points. So here $f' = 6x^2 - 30x + 24 = 6(x-1)(x-4)$ and $f' = 0$ when $x = 1, 4$ but $x = 4$ is outside the interval

$$f(0) = 0, \quad f(1) = 11(max), \quad f(3) = -9(min). \tag{3}$$

3. State Rolles Theorem. Verify Rolles Theorem for the following

$$\begin{array}{cc} (i) & f(x) = x^2 - 2x + 3 \quad \text{on} \quad [0, 2] \\ (ii) & f(x) = x^4 - 2x^2 + 1 \quad \text{on} \quad [-2, 2] \end{array}$$

Soln. Rolles theorem states that if $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there exists a c (at least one) in (a, b) such that

$$f'(c) = 0. \tag{4}$$

Soln (i) Since f is continuous on $[0,2]$ and differentiable on $(0,2)$ and $f(0) = f(2) = 3$, Rolles thm. applies. Now $f' = 2x - 2$ and $f' = 0$ when $x = 1$ (this is c) which is inside $(0,2)$.

Soln (ii) Since f is continuous on $[-2,2]$ and differentiable on $(-2,2)$ and $f(-2) = f(2) = 9$, Rolles thm. applies. Now, $f' = 4x^3 - 4x$ and $f' = 0$ when $x = 0, -1$ and 1 (these are c) and all three are inside $(-2,2)$.

4. State the Mean Value Theorem. Verify the Mean Value Theorem for the following

$$\begin{array}{ll} (i) & f(x) = x^3 - x \quad \text{on} \quad [0,2] \\ (ii) & f(x) = \dfrac{x}{x+2} \quad \text{on} \quad [1,10] \end{array}$$

Soln. The MVT states that if $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then there exists a c (at least one) in (a,b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{5}$$

Soln (i) We have $\dfrac{f(2) - f(0)}{2 - 0} = \dfrac{6}{2} = 3$. Also $f' = 3x^2 - 1$ and $f'(c) = 3c^2 - 1 = 3$ gives $c = 2/\sqrt{3}$

Soln (ii) We have $\dfrac{f(10) - f(1)}{10 - 1} = \dfrac{1/2}{9} = 1/18$. Also $f'(x) = \dfrac{2}{(x+2)^2}$ and

$$f'(c) = \frac{2}{(c+2)^2} = \frac{1}{18} \quad \Rightarrow \quad c = -8, 4$$

from which we choose $c = 4$.

5. If $y = x(x-4)^3$ calculate the following

(i) The critical numbers

(ii) When y is increasing and decreasing.

(iii) Determine whether any of the critical numbers are minimum or maximum.

(iv) When y is concave up and down and determine the points of inflection.

(v) Then sketch the curve.

Soln

$$y' = (x-4)^3 + 3x(x-4)^2 = 4(x-1)(x-4)^2$$

and $y' = 0$ when $x = 1, 4$ (Critical numbers)

$$y'' = 4(x-2)^2 + 8(x-1)(x-4) = 12(x-2)(x-4)$$

and $y'' = 0$ when $x = 2, 4$ (possible PI's.)

x		1		2		4	
$x-1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-4)^2$	$+$	$+$	$+$	$+$	$+$	0	$+$
$(x-1)(x-4)^2$	$-$	0	$+$	$+$	$+$	0	$+$
slope	\	__	/	/	/	—	/
$x-2$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	0	$+$
$(x-2)(x-4)$	$+$	$+$	$+$	0	$-$	0	$+$
h/v	⌣	⌣	⌣	PI	⌢	PI	⌣

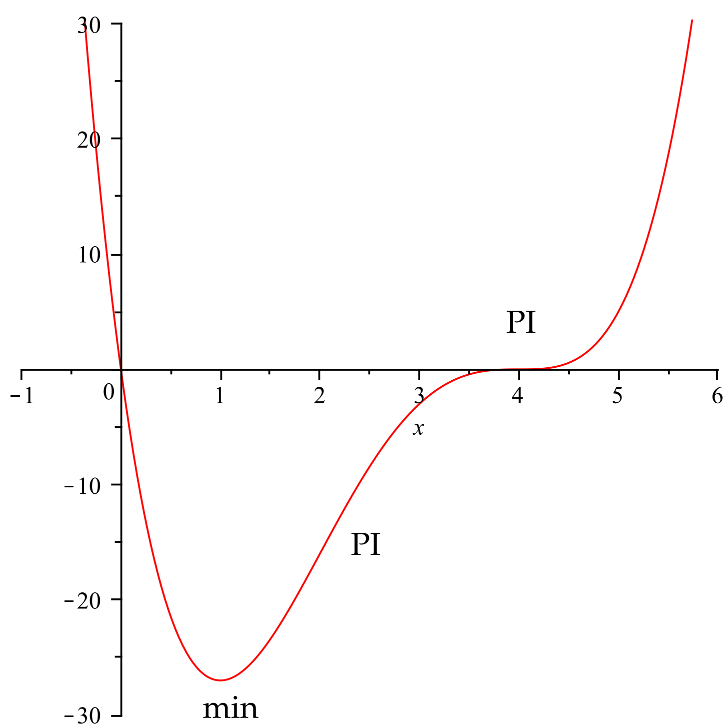
critical numbers $x = 1, 4$

increasing $(1,4)$, $(4,\infty)$ decreasing $(-\infty,1)$

min$(1,-27)$ max - none

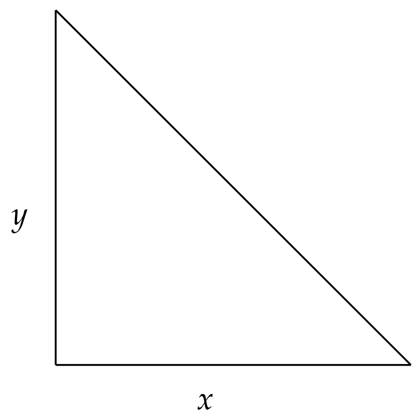
concave up $(\infty,2)$, $(4,\infty)$ concave down $(2,4)$

PI$(2,-16),(4,0)$



6. A ladder 13 feet long is resting against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 ft/sec. At rate is the tip of the ladder moving down the wall when the base of the ladder is 5 ft away from the wall?

Soln.



What we know: $\frac{dx}{dt} = 2$

What we want: $\frac{dy}{dt}$ when $x = 5$

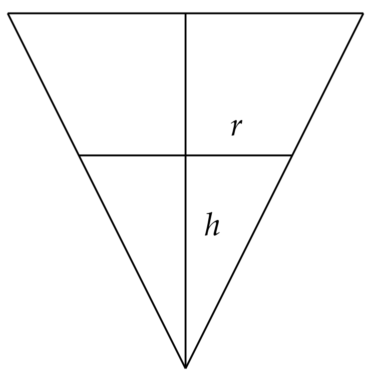
Relate rates: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

so $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$

When $x = 4, y = 12$ so $\frac{dy}{dt} = -\frac{5}{12} \cdot 2 = -\frac{5}{6}$ ft/sec.

7. A paper cup in the shape of an inverted cone with height 10 cm and a base of radius 3 cm, is being filled at a rate of 2 cm^3/min. Find the rate of change in the height of the water when the height of the water is 5 cm.

Soln.



What we know: $\frac{dV}{dt} = 2$

What we want: $\frac{dh}{dt}$ when $h = 5$

Relate variables: $V = \frac{1}{3}\pi r^2 h.$

We also have similar triangles so $\frac{h}{10} = \frac{r}{3}$

so $V = \frac{3\pi h^3}{100}$

Relate rates: $\frac{dV}{dt} = \frac{9\pi h^2}{100}\frac{dh}{dt} = 0$ so $\frac{dh}{dt} = \frac{100}{9\pi h^2}\frac{dV}{dt}$

When $h = 5$ $\frac{dh}{dt} = \frac{100}{9\pi 5^2} \cdot 2 = \frac{8}{9\pi}$ cm/min.