Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise A, Question 1

Question:

Find the binomial expansion of the following up to and including the terms in x^3. State the range values of x for which these expansions are valid.

(a) $(1 + 2x)^3$

(b) $\frac{1}{1 - x}$

(c) $\sqrt{(1 + x)}$

(d) $\frac{1}{(1 + 2x)^3}$

(e) $\sqrt[3]{(1 - 3x)}$

(f) $(1 - 10x)^{\frac{3}{2}}$

(g) $\left(1 + \frac{x}{4}\right)^{-4}$

(h) $\frac{1}{(1 + 2x^2)}$

Solution:

(a) $(1 + 2x)^3$ Use expansion with $n = 3$ and x replaced with $2x$

$= 1 + 3\left(2x\right) + \frac{3 \times 2 \times (2x)^2}{2!} + \frac{3 \times 2 \times 1 \times (2x)^3}{3!} + \frac{3 \times 2 \times 1 \times 0 \times (2x)^4}{4!} + \ldots$

$= 1 + 6x + 12x^2 + 8x^3 + 0x^4$ All terms after $0x^4$ will also be zero

$= 1 + 6x + 12x^2 + 8x^3$

Expansion is finite and exact. Valid for all values of x.

(b) $\frac{1}{1-x}$ Write in index form

$= (1-x)^{-1}$ Use expansion with $n = -1$ and x replaced with $-x$

$= 1 + \left(-1\right)\left(-x\right) + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \ldots$

$= 1 + 1x + 1x^2 + 1x^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots$

Expansion is infinite. Valid when $|-x| < 1 \Rightarrow |x| < 1$.

(c) $\sqrt{1+x}$ Write in index form

$= (1+x)^{\frac{1}{2}}$ Use expansion with $n = \frac{1}{2}$ and x replaced with x

$= 1 + \left(\frac{1}{2}\right)\left(x\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^3}{3!} + \ldots$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$

Expansion is infinite. Valid when $|x| < 1$.

(d) $\frac{1}{(1+2x)^3}$ Write in index form

$= (1+2x)^{-3}$ Use expansion with $n = -3$ and x replaced with $2x$

$= 1 + \left(-3\right)\left(2x\right) + \frac{(-3)(-4)(2x)^2}{2!} + \frac{(-3)(-4)(-5)(2x)^3}{3!} + \ldots$

$= 1 - 6x + 24x^2 - 80x^3 + \ldots$

Expansion is infinite. Valid when $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$.

(e) $\sqrt[3]{(1-3x)}$ Write in index form

$= (1-3x)^{\frac{1}{3}}$ Use expansion with $n = \frac{1}{3}$ and x replaced with $-3x$

$$= 1 + \left(\frac{1}{3}\right)\left(-3x\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-3x\right)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-3x\right)^3}{3!} + \ldots$$

$$= 1 - x - x^2 - \frac{5}{3}x^3 + \ldots$$

Expansion is infinite. Valid when $|-3x| < 1 \Rightarrow |x| < \frac{1}{3}$.

(f) $(1-10x)^{\frac{3}{2}}$ Use expansion with $n = \frac{3}{2}$ and x replaced with $-10x$

$$= 1 + \left(\frac{3}{2}\right)\left(-10x\right) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-10x\right)^2}{2!} + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-10x\right)^3}{3!} + \ldots$$

$$= 1 - 15x + \frac{3}{8} \times 100x^2 - \frac{1}{16} \times \left(-1000x^3\right) + \ldots$$

$$= 1 - 15x + \frac{75}{2}x^2 + \frac{125}{2}x^3 + \ldots$$

Expansion is infinite. Valid when $|-10x| < 1 \Rightarrow |x| < \frac{1}{10}$.

(g) $\left(1+\frac{x}{4}\right)^{-4}$ Use expansion with $n = -4$ and x replaced with $\frac{x}{4}$

$$= 1 + \left(-4\right)\left(\frac{x}{4}\right) + \frac{(-4)(-5)}{2!}\left(\frac{x}{4}\right)^2 + \frac{(-4)(-5)(-6)}{3!}\left(\frac{x}{4}\right)^3 + \ldots$$

$$= 1 - x + 10 \times \frac{x^2}{16} - 20 \times \frac{x^3}{64} + \ldots$$

$$= 1 - x + \frac{5}{8}x^2 - \frac{5}{16}x^3 + \ldots$$

Expansion is infinite. Valid when $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$.

(h) $\frac{1}{1+2x^2}$ Write in index form

$= (1+2x^2)^{-1}$ Use expansion with $n = -1$ and x replaced with $2x^2$

$$= 1 + (-1)(2x^2) + \frac{(-1)(-2)(2x^2)^2}{2!} + \ldots$$

$$= 1 - 2x^2 + \ldots$$

Expansion is infinite. Valid when $|2x^2| < 1 \Rightarrow |x| < \frac{1}{\sqrt{2}}$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise A, Question 2

Question:

By first writing $\frac{(1+x)}{(1-2x)}$ as $(1+x)(1-2x)^{-1}$ show that the cubic approximation to $\frac{(1+x)}{(1-2x)}$ is $1+3x+6x^2+12x^3$. State the range of values of x for which this expansion is valid.

Solution:

$\frac{1+x}{1-2x} = \left(1+x\right)(1-2x)^{-1}$ Expand $(1-2x)^{-1}$ using binomial expansion

$$= \left(1+x\right)\left[1+\left(-1\right)\left(-2x\right)+\frac{(-1)(-2)(-2x)^2}{2!}+\frac{(-1)(-2)(-3)(-2x)^3}{3!}+\ldots\right]$$

$= (1+x)(1+2x+4x^2+8x^3+\ldots)$ Multiply out

$= 1+2x+4x^2+8x^3+\ldots+x+2x^2+4x^3+8x^4+\ldots$ Add like terms

$= 1+3x+6x^2+12x^3+\ldots$

$(1-2x)^{-1}$ is only valid when $|-2x|<1 \Rightarrow |x|<\frac{1}{2}$

So expansion of $\frac{1+x}{1-2x}$ is only valid when $|x|<\frac{1}{2}$.

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise A, Question 3

Question:

Find the binomial expansion of $\sqrt{(1+3x)}$ in ascending powers of x up to and including the term in x^3. By substituting $x = 0.01$ in the expansion, find an approximation to $\sqrt{103}$. By comparing it with the exact value, comment on the accuracy of your approximation.

Solution:

$$\sqrt{(1+3x)} = (1+3x)^{\frac{1}{2}}$$

$$= 1 + \left(\frac{1}{2}\right)(3x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(3x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(3x)^3}{3!} + \ldots$$

$$= 1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3 + \ldots$$

This expansion is valid if $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

Substitute $x = 0.01$ (OK, as $|x| < \frac{1}{3}$) into both sides to give

$$\sqrt{1 + 3 \times 0.01} \simeq 1 + \frac{3}{2} \times 0.01 - \frac{9}{8} \times 0.01^2 + \frac{27}{16} \times 0.01^3$$

$$\sqrt{1.03} \simeq 1 + 0.015 - 0.0001125 + 0.0000016875$$

$$\sqrt{\frac{103}{100}} \simeq 1.014889188 \qquad \left(\sqrt{\frac{103}{100}} = \frac{\sqrt{103}}{\sqrt{100}} = \frac{\sqrt{103}}{10}\right)$$

$$\frac{\sqrt{103}}{10} \simeq 1.014889188 \qquad (\times 10)$$

$$\sqrt{103} \simeq 10.14889188$$

Using a calculator

$\sqrt{103} = 10.14889157$

Hence approximation correct to 6 d.p.

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise A, Question 4

Question:

In the expansion of $(1 + ax)^{-\frac{1}{2}}$ the coefficient of x^2 is 24. Find possible values of the constant a and the corresponding term in x^3.

Solution:

$$(1 + ax)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)\left(ax\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(ax)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(ax)^3}{3!} + \ldots$$

$$= 1 - \frac{1}{2}ax + \frac{3}{8}a^2x^2 - \frac{5}{16}a^3x^3 + \ldots$$

This expansion is valid if $|ax| < 1 \Rightarrow |x| < \frac{1}{a}$.

If coefficient of x^2 is 24 then

$\frac{3}{8}a^2 = 24$

$a^2 = 64$

$a = \pm 8$

Term in x^3 is

$-\frac{5}{16}a^3x^3 = -\frac{5}{16}(\pm 8)^3x^3 = \pm 160x^3$

If $a = 8$, term in x^3 is $-160x^3$

If $a = -8$, term in x^3 is $+160x^3$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise A, Question 5

Question:

Show that if x is small, the expression $\sqrt{\left(\frac{1+x}{1-x}\right)}$ is approximated by $1 + x + \frac{1}{2}x^2$.

Solution:

$$\sqrt{\frac{1+x}{1-x}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$$

$$= (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$ Expand using the binomial expansion

$$= \left[1 + \left(\frac{1}{2}\right)(x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x)^2}{2!} + \ldots\right]\left[1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!} + \ldots\right]$$

$$= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right)$$

$$= 1\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right) + \frac{1}{2}x\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right) - \frac{1}{8}x^2\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \ldots\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 + \ldots$$ Add like terms

$$= 1 + x + \frac{1}{2}x^2 + \ldots$$

Hence $\sqrt{\frac{1+x}{1-x}} \simeq 1 + x + \frac{1}{2}x^2$

If terms larger than or equal to x^3 are ignored.

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise A, Question 6

Question:

Find the first four terms in the expansion of $(1-3x)^{\frac{3}{2}}$. By substituting in a suitable value of x, find an approximation to $97^{\frac{3}{2}}$.

Solution:

$$(1-3x)^{\frac{3}{2}} = 1 + \left(\frac{3}{2}\right)(-3x) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)(-3x)^2}{2!} + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-3x)^3}{3!} + \ldots$$

$$= 1 - \frac{9x}{2} + \frac{27x^2}{8} + \frac{27x^3}{16} + \ldots$$

Expansion is valid if $|-3x| < 1 \Rightarrow |x| < \frac{1}{3}$.

Substitute $x = 0.01$ into both sides of expansion to give

$$(1 - 3 \times 0.01)^{\frac{3}{2}} = 1 - \frac{9 \times 0.01}{2} + \frac{27 \times (0.01)^2}{8} + \frac{27 \times (0.01)^3}{16} + \ldots$$

$$(0.97)^{\frac{3}{2}} \simeq 1 - 0.045 + 0.0003375 + 0.000001687$$

$$(0.97)^{\frac{3}{2}} \simeq 0.955339187$$

$$\left(\frac{97}{100}\right)^{\frac{3}{2}} \simeq 0.955339187, \quad \left[\left(\frac{97}{100}\right)^{\frac{3}{2}} = \frac{97^{\frac{3}{2}}}{100^{\frac{3}{2}}} = \frac{97^{\frac{3}{2}}}{(\sqrt{100})^3} = \frac{97^{\frac{3}{2}}}{1000}\right]$$

$$\frac{97^{\frac{3}{2}}}{1000} \simeq 0.955339187 \quad (\times 1000)$$

$$97^{\frac{3}{2}} \simeq 955.339187$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise B, Question 1

Question:

Find the binomial expansions of the following in ascending powers of x as far as the term in x^3. State the range of values of x for which the expansions are valid.

(a) $\sqrt{(4 + 2x)}$

(b) $\frac{1}{2 + x}$

(c) $\frac{1}{(4 - x)^2}$

(d) $\sqrt{(9 + x)}$

(e) $\frac{1}{\sqrt{(2 + x)}}$

(f) $\frac{5}{3 + 2x}$

(g) $\frac{1 + x}{2 + x}$

(h) $\sqrt{\left(\frac{2 + x}{1 - x}\right)}$

Solution:

(a) $\sqrt{(4 + 2x)}$ Write in index form.

$= (4 + 2x)^{\frac{1}{2}}$ Take out a factor of 4

$= \left[4\left(1 + \frac{2x}{4}\right)\right]^{\frac{1}{2}}$ Remember to put the 4 to the power $\frac{1}{2}$

$= 4^{\frac{1}{2}}\left(1 + \frac{x}{2}\right)^{\frac{1}{2}}$ $4^{\frac{1}{2}} = 2$

$= 2\left(1 + \frac{x}{2}\right)^{\frac{1}{2}}$ Use the binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{2}$

$$= 2\left[1 + \left(\frac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{(\frac{1}{2})(-\frac{1}{2})(\frac{x}{2})^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(\frac{x}{2})^3}{3!} + \ldots\right]$$

$$= 2\left(1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} + \ldots\right)$$ Multiply by the 2

$$= 2 + \frac{x}{2} - \frac{x^2}{16} + \frac{x^3}{64}$$

Valid if $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(b) $\frac{1}{2+x}$ Write in index form

$= (2+x)^{-1}$ Take out a factor of 2

$= \left[2\left(1 + \frac{x}{2}\right)\right]^{-1}$ Remember to put 2 to the power -1

$= 2^{-1}\left(1 + \frac{x}{2}\right)^{-1}$, $2^{-1} = \frac{1}{2}$. Use the binomial expansion with $n = -1$ and $x = \frac{x}{2}$

$$= \frac{1}{2}\left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$= \frac{1}{2}\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right)$ Multiply by the $\frac{1}{2}$

$= \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$

Valid if $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(c) $\frac{1}{(4-x)^2}$ Write in index form

$= (4-x)^{-2}$ Take 4 out as a factor

$= \left[4\left(1 - \frac{x}{4}\right)\right]^{-2}$

$= 4^{-2}\left(1 - \frac{x}{4}\right)^{-2}$, $4^{-2} = \frac{1}{16}$. Use the binomial expansion with $n = -2$ and $x = \frac{x}{4}$

$= \frac{1}{16}\left[1 + \left(-2\right)\left(-\frac{x}{4}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{x}{4}\right)^3 + \ldots\right]$

$= \frac{1}{16}\left(1 + \frac{x}{2} + \frac{3x^2}{16} + \frac{x^3}{16} + \ldots\right)$ Multiply by $\frac{1}{16}$

$= \frac{1}{16} + \frac{x}{32} + \frac{3x^2}{256} + \frac{x^3}{256}$

Valid for $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$

(d) $\sqrt{9+x}$ Write in index form

$= (9+x)^{\frac{1}{2}}$ Take 9 out as a factor

$= \left[9\left(1 + \frac{x}{9}\right)\right]^{\frac{1}{2}}$

$= 9^{\frac{1}{2}}\left(1 + \frac{x}{9}\right)^{\frac{1}{2}}$, $9^{\frac{1}{2}} = 3$. Use binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{9}$

$$= 3\left[1 + \left(\frac{1}{2}\right)\left(\frac{x}{9}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{9}\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{9}\right)^3 + \ldots\right]$$

$$= 3\left(1 + \frac{x}{18} - \frac{x^2}{648} + \frac{x^3}{11664} + \ldots\right)$$ Multiply by 3

$$= 3 + \frac{x}{6} - \frac{x^2}{216} + \frac{x^3}{3888}$$

Valid for $\left|\frac{x}{9}\right| < 1 \Rightarrow |x| < 9$

(e) $\frac{1}{\sqrt{2+x}}$ Write in index form

$= (2 + x)^{-\frac{1}{2}}$ Take out a factor of 2

$$= \left[2\left(1 + \frac{x}{2}\right)\right]^{-\frac{1}{2}}$$

$= 2^{-\frac{1}{2}}\left(1 + \frac{x}{2}\right)^{-\frac{1}{2}}$, $2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$. Use binomial expansion with $n = -\frac{1}{2}$ and $x = \frac{x}{2}$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2 +\right.$$

$$\left.\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\ \ldots\right]$$

$$=\frac{1}{\sqrt{2}}\left(1-\frac{x}{4}+\frac{3x^2}{32}-\frac{5x^3}{128}+\ \ldots\right)$$ Multiply by $\frac{1}{\sqrt{2}}$

$$=\frac{1}{\sqrt{2}}-\frac{x}{4\sqrt{2}}+\frac{3x^2}{32\sqrt{2}}-\frac{5x^3}{128\sqrt{2}}+\ \ldots$$ Rationalise surds

$$=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}x}{8}+\frac{3\sqrt{2}x^2}{64}-\frac{52x^3}{256}$$

Valid if $\left|\frac{x}{2}\right|<1 \Rightarrow |x|<2$

(f) $\frac{5}{3+2x}$ Write in index form

$=5(3+2x)^{-1}$ Take out a factor of 3

$=5\left[3\left(1+\frac{2x}{3}\right)\right]^{-1}$

$=5\times3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$, $3^{-1}=\frac{1}{3}$. Use binomial expansion with $n=-1$ and $x=\frac{2x}{3}$

$$=\frac{5}{3}\left[1+\left(-1\right)\left(\frac{2x}{3}\right)+\frac{(-1)(-2)}{2!}\left(\frac{2x}{3}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{2x}{3}\right)^3+\ \ldots\right]$$

$$=\frac{5}{3}\left(1-\frac{2x}{3}+\frac{4x^2}{9}-\frac{8x^3}{27}+\ \ldots\right)$$ Multiply by $\frac{5}{3}$

$$=\frac{5}{3}-\frac{10x}{9}+\frac{20x^2}{27}-\frac{40x^3}{81}$$

Valid if $\left|\frac{2x}{3}\right|<1 \Rightarrow |x|<\frac{3}{2}$

(g) $\frac{1+x}{2+x}$ Write $\frac{1}{2+x}$ in index form

$= (1+x)(2+x)^{-1}$ Take out a factor of 2

$= \left(1+x\right)\left[2\left(1+\frac{x}{2}\right)\right]^{-1}$

$= \left(1+x\right)2^{-1}\left(1+\frac{x}{2}\right)^{-1}$ Expand $\left(1+\frac{x}{2}\right)^{-1}$ using the binomial expansion

$= \left(1+x\right)\frac{1}{2}\left[1+\left(-1\right)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right]$

$= \left(1+x\right)\frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$ Multiply $\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\ldots\right)$ by $\frac{1}{2}$

$= \left(1+x\right)\left(\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots\right)$ Multiply your answer by $(1+x)$

$= 1\left(\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots\right)+x\left(\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots\right)$

$= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\ldots+\frac{x}{2}-\frac{x^2}{4}+\frac{x^3}{8}+\ldots$ Collect like terms

$= \frac{1}{2}+\frac{1}{4}x-\frac{1}{8}x^2+\frac{1}{16}x^3$

Valid if $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(h) $\sqrt{\frac{2+x}{1-x}}$

$= (2+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ Put both in index form

$= 2^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$ Expand both using the binomial expansion

$$= \sqrt{2}\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right]\left[1+\left(-\frac{1}{2}\right)\left(-x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^3}{3!}+\ldots\right]$$

$= \sqrt{2}\left(1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 + \ldots\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\right)$ Multiply out

$= \sqrt{2}\left[1\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\right) + \frac{1}{4}x\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\right) - \frac{1}{32}x^2\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\right) + \frac{1}{128}x^3\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\right) \ldots\right]$

$= \sqrt{2}\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{3}{32}x^3 - \frac{1}{32}x^2 - \frac{1}{64}x^3 + \frac{1}{128}x^3 + \ldots\right)$ Collect like terms

$= \sqrt{2}\left(1 + \frac{3}{4}x + \frac{15}{32}x^2 + \frac{51}{128}x^3 + \ldots\right)$ Multiply by $\sqrt{2}$

$= \sqrt{2} + \frac{3\sqrt{2}}{4}x + \frac{15\sqrt{2}}{32}x^2 + \frac{51\sqrt{2}}{128}x^3$

Valid if $\left|\frac{x}{2}\right| < 1$ and $|-x| < 1 \Rightarrow |x| < 1$ for both to be valid

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise B, Question 2

Question:

Prove that if x is sufficiently small, $\frac{3+2x-x^2}{4-x}$ may be approximated by $\frac{3}{4}+\frac{11}{16}x-\frac{5}{64}x^2$. What does 'sufficiently small' mean in this question?

Solution:

$\frac{3+2x-x^2}{4-x} \equiv \left(3+2x-x^2\right)\left(4-x\right)^{-1}$ Write $\frac{1}{4-x}$ as $(4-x)^{-1}$

$= \left(3+2x-x^2\right)\left[4\left(1-\frac{x}{4}\right)\right]^{-1}$ Take out a factor of 4

$= \left(3+2x-x^2\right)\frac{1}{4}\left(1-\frac{x}{4}\right)^{-1}$ Expand $\left(1-\frac{x}{4}\right)^{-1}$ using the binomial expansion

$= \left(3+2x-x^2\right)\frac{1}{4}\left[1+\left(-1\right)\left(-\frac{x}{4}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{4}\right)^2+\ldots\right]$ Ignore terms higher than x^2

$= \left(3+2x-x^2\right)\frac{1}{4}\left(1+\frac{x}{4}+\frac{x^2}{16}+\ldots\right)$ Multiply expansion by $\frac{1}{4}$

$= \left(3+2x-x^2\right)\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)$ Multiply result by $(3+2x-x^2)$

$= 3\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)+2x\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)-x^2\left(\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\ldots\right)$

$= \frac{3}{4} + \frac{3}{16}x + \frac{3}{64}x^2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{4}x^2 + \ldots$ Ignore any terms bigger than x^2

$= \frac{3}{4} + \frac{11}{16}x - \frac{5}{64}x^2$

Expansion is valid if $\left| \frac{-x}{4} \right| < 1 \Rightarrow |x| < 4$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise B, Question 3

Question:

Find the first four terms in the expansion of $\sqrt{(4-x)}$. By substituting $x=\frac{1}{9}$, find a fraction that is an approximation to $\sqrt{35}$. By comparing this to the exact value, state the degree of accuracy of your approximation.

Solution:

$$\sqrt{(4-x)} = (4-x)^{\frac{1}{2}}$$

$$= \left[4\left(1-\frac{x}{4}\right)\right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$$

$$= 2\left[1+\left(\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3+\ldots\right]$$

$$= 2\left(1-\frac{x}{8}-\frac{x^2}{128}-\frac{x^3}{1024}+\ldots\right)$$

$$= 2-\frac{x}{4}-\frac{x^2}{64}-\frac{x^3}{512}+\ldots$$

Valid for $\left|-\frac{x}{4}\right|<1 \Rightarrow |x|<4$

Substitute $x=\frac{1}{9}$ into both sides of the expansion:

$$\sqrt{\left(4-\frac{1}{9}\right)} \simeq 2-\frac{\frac{1}{9}}{4}-\frac{\left(\frac{1}{9}\right)^2}{64}-\frac{\left(\frac{1}{9}\right)^3}{512}$$

$$\sqrt{\frac{35}{9}} \simeq 2-\frac{1}{36}-\frac{1}{5184}-\frac{1}{373248}$$

$$\frac{\sqrt{35}}{3} \simeq \frac{736055}{373248}$$

$$\sqrt{35} \simeq 3\times\frac{736055}{373248}=\frac{736055}{124416}=5.916079 \mid 925$$

By calculator $\sqrt{35} = 5.916079 \mid 783$
Fraction accurate to 6 decimal places

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise B, Question 4

Question:

The expansion of $(a + bx)^{-2}$ may be approximated by $\frac{1}{4} + \frac{1}{4}x + cx^2$. Find the values of the constants a, b and c.

Solution:

$(a + bx)^{-2} = [a (1 + \frac{bx}{a})]^{-2}$ Take out a factor of a

$= a^{-2} (1 + \frac{bx}{a})^{-2}$

$= \frac{1}{a^2} (1 + \frac{bx}{a})^{-2}$

$= \frac{1}{a^2} [1 + (-2) (\frac{bx}{a}) + \frac{(-2) (-3)}{2 !} (\frac{bx}{a})^2 + \ldots]$

$= \frac{1}{a^2} - \frac{2bx}{a^3} + \frac{3b^2x^2}{a^4} + \ldots$

Compare this to $\frac{1}{4} + \frac{1}{4}x + cx^2$

Comparing constant terms: $\frac{1}{a^2} = \frac{1}{4}$

$\Rightarrow a^2 = 4 \quad (\sqrt{})$

$\Rightarrow a = \pm 2$

Comparing terms in x: $\frac{-2b}{a^3} = \frac{1}{4}$

$\Rightarrow b = \frac{a^3}{-8}$ Substitute $a = \pm 2$

$\Rightarrow b = \frac{(\pm 2)^3}{-8}$

$\Rightarrow b = \pm 1$

Compare terms in x^2: $c = \frac{3b^2}{a^4}$ Substitute $a^4 = 16$, $b^2 = 1$

$\Rightarrow c = \frac{3 \times 1}{16}$

$\Rightarrow \quad c = \frac{3}{16}$

Hence $a = \pm 2$, $b \pm 1$, $c = \frac{3}{16}$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise C, Question 1

Question:

(a) Express $\frac{8x+4}{(1-x)(2+x)}$ as partial fractions.

(b) Hence or otherwise expand $\frac{8x+4}{(1-x)(2+x)}$ in ascending powers of x as far as the term in x^2.

(c) State the set of values of x for which the expansion is valid.

Solution:

(a) Let $\frac{8x+4}{(1-x)(2+x)} \equiv \frac{A}{(1-x)} + \frac{B}{(2+x)} \equiv \frac{A(2+x)+B(1-x)}{(1-x)(2+x)}$

Set the numerators equal: $8x+4 \equiv A(2+x)+B(1-x)$

Substitute $x=1$: $8 \times 1 + 4 = A \times 3 + B \times 0$

$\Rightarrow \quad 12 = 3A$

$\Rightarrow \quad A = 4$

Substitute $x = -2$: $8 \times (-2) + 4 = A \times 0 + B \times 3$

$\Rightarrow \quad -12 = 3B$

$\Rightarrow \quad B = -4$

Hence $\frac{8x+4}{(1-x)(2+x)} \equiv \frac{4}{(1-x)} - \frac{4}{(2+x)}$

(b) $\frac{4}{(1-x)} = 4(1-x)^{-1}$

$= 4\left[1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \ldots\right]$

$= 4(1 + x + x^2 + \ldots)$

$= 4 + 4x + 4x^2 + \ldots$

$\frac{4}{(2+x)} = 4(2+x)^{-1}$

$$= 4\left[2\left(1+\frac{x}{2}\right)\right]^{-1}$$

$$= 4 \times 2^{-1}\left(1+\frac{x}{2}\right)^{-1}$$

$$= 4 \times \frac{1}{2} \times \left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \ldots\right]$$

$$= 2\left(1 - \frac{x}{2} + \frac{x^2}{4} + \ldots\right)$$

$$= 2 - x + \frac{1}{2}x^2 + \ldots$$

Therefore

$$\frac{8x+4}{(1-x)(2+x)} \equiv \frac{4}{(1-x)} - \frac{4}{(2+x)}$$

$$= \left(4 + 4x + 4x^2 + \ldots\right) - \left(2 - x + \frac{1}{2}x^2 + \ldots\right)$$

$$= 2 + 5x + \frac{7x^2}{2}$$

(c) $\frac{4}{(1-x)}$ is valid for $|x| < 1$

$\frac{4}{(2+x)}$ is valid for $|x| < 2$

Both are valid when $|x| < 1$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise C, Question 2

Question:

(a) Express $\frac{-2x}{(2+x)^2}$ as a partial fraction.

(b) Hence prove that $\frac{-2x}{(2+x)^2}$ can be expressed in the form $0-\frac{1}{2}x+Bx^2+Cx^3$ where constants B and C are to be determined.

(c) State the set of values of x for which the expansion is valid.

Solution:

(a) Let $\frac{-2x}{(2+x)^2} \equiv \frac{A}{(2+x)} + \frac{B}{(2+x)^2} \equiv \frac{A(2+x)+B}{(2+x)^2}$

Set the numerators equal: $-2x \equiv A(2+x)+B$

Substitute $x=-2$: $4 = A \times 0 + B \Rightarrow B = 4$

Equate terms in x: $-2 = A \Rightarrow A = -2$

Hence $\frac{-2x}{(2+x)^2} \equiv \frac{-2}{(2+x)} + \frac{4}{(2+x)^2}$

(b) $\frac{-2}{2+x} = -2(2+x)^{-1}$

$$= -2\left[2\left(1+\frac{x}{2}\right)\right]^{-1}$$

$$= -2 \times 2^{-1} \times \left(1+\frac{x}{2}\right)^{-1}$$

$$= -1 \times \left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= -1 \times \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right)$$

$$= -1 + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} + \ldots$$

$$\frac{4}{(2+x)^2} = 4(2+x)^{-2}$$

$$= 4\left[2\left(1+\frac{x}{2}\right)\right]^{-2}$$

$$= 4 \times 2^{-2} \times \left(1+\frac{x}{2}\right)^{-2}$$

$$= 1 \times \left[1 + \left(-2\right)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= 1 \times \left(1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \ldots\right)$$

$$= 1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \ldots$$

Hence

$$\frac{-2x}{(2+x)^2} \equiv \frac{-2}{(2+x)} + \frac{4}{(2+x)^2}$$

$$= -1 + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} + 1 - x + \frac{3x^2}{4} - \frac{x^3}{2} + \ldots$$

$$= 0 - \frac{1}{2}x + \frac{1}{2}x^2 - \frac{3}{8}x^3$$

Hence $B = \frac{1}{2}$, (coefficient of x^2) and $C = -\frac{3}{8}$, (coefficients of x^3)

(c) $\frac{-2}{(2+x)}$ is valid for $|x| < 2$

$\frac{4}{(2+x)^2}$ is valid for $|x| < 2$

Hence whole expression is valid $|x| < 2$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise C, Question 3

Question:

(a) Express $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ as a partial fraction.

(b) Hence or otherwise expand $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ in ascending powers of x as far as the term in x^3.

(c) State the set of values of x for which the expansion is valid.

Solution:

(a) Let $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{A}{(1+x)} + \frac{B}{(1-x)} + \frac{C}{(2+x)}$

$$\equiv \frac{A(1-x)(2+x)+B(1+x)(2+x)+C(1+x)(1-x)}{(1+x)(1-x)(2+x)}$$

Set the numerators equal:

$$6+7x+5x^2 \equiv A\left(1-x\right)\left(2+x\right)+B\left(1+x\right)\left(2+x\right)+C\left(1+x\right)\left(1-x\right)$$

Substitute $x=1$: $6+7+5=A\times 0+B\times 2\times 3+C\times 0$

$\Rightarrow$ $18=6B$

$\Rightarrow$ $B=3$

Substitute $x=-1$: $6-7+5=A\times 2\times 1+B\times 0+C\times 0$

$\Rightarrow$ $4=2A$

$\Rightarrow$ $A=2$

Substitute $x=-2$: $6-14+20=A\times 0+B\times 0+C\times(-1)\times 3$

$\Rightarrow$ $12=-3C$

$\Rightarrow$ $C=-4$

Hence $\frac{6 + 7x + 5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{(1+x)} + \frac{3}{(1-x)} - \frac{4}{(2+x)}$

(b) $\frac{2}{1+x} = 2(1+x)^{-1}$

$$= 2\left[1 + (-1)(x) + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \ldots\right]$$

$$= 2(1 - x + x^2 - x^3 + \ldots)$$

$$\simeq 2 - 2x + 2x^2 - 2x^3 \quad \text{Valid for } |x| < 1$$

$\frac{3}{1-x} = 3(1-x)^{-1}$

$$= 3\left[1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \ldots\right]$$

$$= 3(1 + x + x^2 + x^3 + \ldots)$$

$$\simeq 3 + 3x + 3x^2 + 3x^3 \quad \text{Valid for } |x| < 1$$

$\frac{4}{2+x} = 4(2+x)^{-1}$

$$= 4\left[2\left(1 + \frac{x}{2}\right)\right]^{-1}$$

$$= 4 \times 2^{-1} \times \left(1 + \frac{x}{2}\right)^{-1}$$

$$= 2\left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right)$$

$$\simeq 2 - x + \frac{x^2}{2} - \frac{x^3}{4} \quad \text{Valid for } |x| < 2$$

Hence

$$\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{(1+x)} + \frac{3}{(1-x)} - \frac{4}{(2+x)}$$

$$= \left(2 - 2x + 2x^2 - 2x^3\right) + \left(3 + 3x + 3x^2 + 3x^3\right) - \left(2 - x + \frac{x^2}{2} - \frac{x^3}{4}\right)$$

$$= 2 + 3 - 2 - 2x + 3x + x + 2x^2 + 3x^2 - \frac{x^2}{2} - 2x^3 + 3x^3 + \frac{x^3}{4}$$

$$= 3 + 2x + \frac{9}{2}x^2 + \frac{5}{4}x^3$$

(c) All expansions are valid when $|x| < 1$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 1

Question:

Find binomial expansions of the following in ascending powers of x as far as the term in x^3. State the set of values of x for which the expansion is valid.

(a) $(1-4x)^3$

(b) $\sqrt{(16+x)}$

(c) $\frac{1}{(1-2x)}$

(d) $\frac{4}{2+3x}$

(e) $\frac{4}{\sqrt{(4-x)}}$

(f) $\frac{1+x}{1+3x}$

(g) $\left(\frac{1+x}{1-x}\right)^2$

(h) $\frac{x-3}{(1-x)(1-2x)}$

Solution:

(a) $(1-4x)^3$ Use binomial expansion with $n=3$ and $x=-4x$

$= 1 + \binom{3}{}\left(-4x\right) + \frac{(3)(2)(-4x)^2}{2!} + \frac{(3)(2)(1)(-4x)^3}{3!}$ As $n=3$ expansion is finite and exact

$= 1 - 12x + 48x^2 - 64x^3$ Valid for all x

(b) $\sqrt{16+x}$ Write in index form

$= (16+x)^{\frac{1}{2}}$ Take out a factor of 16

$= \left[16 \left(1 + \frac{x}{16} \right) \right]^{\frac{1}{2}}$

$= 16^{\frac{1}{2}} \left(1 + \frac{x}{16} \right)^{\frac{1}{2}}$ Use binomial expansion with $n = \frac{1}{2}$ and $x = \frac{x}{16}$

$= 4 \left[1 + \frac{1}{2} \left(\frac{x}{16} \right) + \frac{\left(\frac{1}{2} \right) \left(- \frac{1}{2} \right)}{2!} \left(\frac{x}{16} \right)^2 + \frac{\left(\frac{1}{2} \right) \left(- \frac{1}{2} \right) \left(- \frac{3}{2} \right)}{3!} \left(\frac{x}{16} \right)^3 + \ldots \right]$

$= 4 \left(1 + \frac{x}{32} - \frac{x^2}{2048} + \frac{x^3}{65536} + \ldots \right)$ Multiply by 4

$= 4 + \frac{x}{8} - \frac{x^2}{512} + \frac{x^3}{16384} + \ldots$

Valid for $\left| \frac{x}{16} \right| < 1 \Rightarrow |x| < 16$

(c) $\frac{1}{1 - 2x}$ Write in index form

$= (1 - 2x)^{-1}$ Use binomial expansion with $n = -1$ and $x = -2x$

$= 1 + (-1)(-2x) + \frac{(-1)(-2)(-2x)^2}{2!} + \frac{(-1)(-2)(-3)(-2x)^3}{3!} + \ldots$

$= 1 + 2x + 4x^2 + 8x^3 + \ldots$

Valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

(d) $\frac{4}{2 + 3x}$ Write in index form

$= 4(2 + 3x)^{-1}$ Take out a factor of 2

$= 4 \left[2 \left(1 + \frac{3x}{2} \right) \right]^{-1}$

$= 4 \times 2^{-1} \times \left(1 + \frac{3x}{2} \right)^{-1}$ Use binomial expansion with $n = -1$ and

$x = \frac{3x}{2}$

$$= 2\left[1 + \left(-1\right)\left(\frac{3x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{3x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{3x}{2}\right)^3 + \ldots\right]$$

$$= 2\left(1 - \frac{3x}{2} + \frac{9x^2}{4} - \frac{27x^3}{8} + \ldots\right)$$ Multiply by 2

$$= 2 - 3x + \frac{9x^2}{2} - \frac{27x^3}{4} + \ldots$$

Valid for $\left|\frac{3x}{2}\right| < 1 \Rightarrow |x| < \frac{2}{3}$

(e) $\frac{4}{\sqrt{4-x}} = 4\left(\sqrt{4-x}\right)^{-1}$ Write in index form

$= 4(4-x)^{-\frac{1}{2}}$ Take out a factor of 4

$= 4\left[4\left(1 - \frac{x}{4}\right)\right]^{-\frac{1}{2}}$

$= 4 \times 4^{-\frac{1}{2}}\left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}$ Use binomial expansion with $n = -\frac{1}{2}$ and

$x = -\frac{x}{4}$

$$= 4^{\frac{1}{2}}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-\frac{x}{4}\right)^3 + \ldots\right]$$

$$= 2\left(1 + \frac{x}{8} + \frac{3}{128}x^2 + \frac{5}{1024}x^3 + \ldots\right)$$ Multiply by 2

$= 2 + \frac{x}{4} + \frac{3}{64}x^2 + \frac{5}{512}x^3 + \ldots$

Valid $\left| -\frac{x}{4} \right| < 1 \Rightarrow |x| < 4$

(f) $\frac{1+x}{1+3x} = \left(1+x\right)(1+3x)^{-1}$ Write $\frac{1}{1+3x}$ in index form then expand

$= \left(1+x\right)\left[1 + \left(-1\right)\left(3x\right) + \frac{(-1)(-2)(3x)^2}{2!} + \frac{(-1)(-2)(-3)(3x)^3}{3!} + \ldots\right]$

$= (1+x)(1 - 3x + 9x^2 - 27x^3 + \ldots)$ Multiply out

$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3 + \ldots$ Collect like terms

$= 1 - 2x + 6x^2 - 18x^3 + \ldots$

Valid for $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

(g) $\left(\frac{1+x}{1-x}\right)^2 = \frac{(1+x)^2}{(1-x)^2}$ Write in index form

$= (1+x)^2(1-x)^{-2}$ Expand $(1-x)^{-2}$ using binomial expansion

$= \left(1 + 2x + x^2\right)\left[1 + \left(-2\right)\left(-x\right) + \frac{(-2)(-3)(-x)^2}{2!} + \frac{(-2)(-3)(-4)(-x)^3}{3!} + \ldots\right]$

$= (1 + 2x + x^2)(1 + 2x + 3x^2 + 4x^3 + \ldots)$ Multiply out brackets

$= 1 + 2x + 3x^2 + 4x^3 + 2x + 4x^2 + 6x^3 + x^2 + 2x^3 + \ldots$ Collect like terms

$= 1 + 4x + 8x^2 + 12x^3 + \ldots$

Valid for $|x| < 1$

(h) Let $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{A}{(1-x)} + \frac{B}{(1-2x)}$ Put in partial fraction form

$\equiv \frac{A(1-2x) + B(1-x)}{(1-x)(1-2x)}$ Add fractions.

Set the numerators equal: $x - 3 \equiv A(1-2x) + B(1-x)$

Substitute $x = 1$: $1 - 3 = A \times -1 + B \times 0$

$\Rightarrow -2 = -1A$

$\Rightarrow \quad A = 2$

Substitute $x = \frac{1}{2}$: $\quad \frac{1}{2} - 3 = A \times 0 + B \times \frac{1}{2}$

$\Rightarrow \quad -2\frac{1}{2} = \frac{1}{2}B$

$\Rightarrow \quad B = -5$

Hence $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$

$$\frac{2}{(1-x)} = 2(1-x)^{-1}$$

$$= 2\left[1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \ldots\right]$$

$$= 2(1 + x + x^2 + x^3 + \ldots)$$

$$\simeq 2 + 2x + 2x^2 + 2x^3$$

$$\frac{5}{(1-2x)} = 5(1-2x)^{-1}$$

$$= 5\left[1 + (-1)(-2x) + \frac{(-1)(-2)(-2x)^2}{2!} + \frac{(-1)(-2)(-3)(-2x)^3}{3!} + \ldots\right]$$

$$= 5(1 + 2x + 4x^2 + 8x^3 + \ldots)$$

$$\simeq 5 + 10x + 20x^2 + 40x^3$$

Hence $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$

$$\simeq (2 + 2x + 2x^2 + 2x^3) - (5 + 10x + 20x^2 + 40x^3)$$

$$\simeq -3 - 8x - 18x^2 - 38x^3$$

$\frac{2}{1-x}$ is valid for $|x| < 1$

$\frac{5}{1-2x}$ is valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

Both are valid when $|x| < \frac{1}{2}$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 2

Question:

Find the first four terms of the expansion in ascending powers of x of:

$$\left(1-\frac{1}{2}x\right)^{\frac{1}{2}},\ |x|<2$$

and simplify each coefficient. (E) (adapted)

Solution:

$$\left(1-\frac{1}{2}x\right)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)\left(-\frac{1}{2}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}x\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}x\right)^3}{3!}+\ldots$$

$$= 1-\frac{1}{4}x+\left(-\frac{1}{8}\right)\times\left(\frac{1}{4}x^2\right)+\left(\frac{1}{16}\right)\times\left(-\frac{1}{8}x^3\right)+\ldots$$

$$= 1-\frac{1}{4}x-\frac{1}{32}x^2-\frac{1}{128}x^3$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 3

Question:

Show that if x is sufficiently small then $\frac{3}{\sqrt{(4+x)}}$ can be approximated by $\frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$.

Solution:

$\frac{3}{\sqrt{4+x}} = 3(\sqrt{4+x})^{-1}$ Write in index form

$= 3(4+x)^{-\frac{1}{2}}$ Take out a factor of 4

$= 3\left[4\left(1+\frac{x}{4}\right)\right]^{-\frac{1}{2}}$

$= 3 \times 4^{-\frac{1}{2}} \times \left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$ $4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2}$

$= \frac{3}{2} \times \left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!} + \ldots\right]$

$= \frac{3}{2}\left(1 - \frac{x}{8} + \frac{3}{128}x^2 + \ldots\right)$ Multiply by $\frac{3}{2}$

$= \frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2 + \ldots$

$= \frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$ If terms higher than x^2 are ignored

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 4

Question:

Given that $|x| < 4$, find, in ascending powers of x up to and including the term in x^3, the series expansion of:

(a) $(4-x)^{\frac{1}{2}}$

(b) $(4-x)^{\frac{1}{2}}(1+2x)$ **E** (adapted)

Solution:

(a) $(4-x)^{\frac{1}{2}}$ Take out a factor of 4

$$= \left[4\left(1-\frac{x}{4}\right)\right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$$ Use binomial expansion with $n=\frac{1}{2}$ and $x=-\frac{x}{4}$

$$= 2\left[1+\left(\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)^2}{2!}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{x}{4}\right)^3}{3!}+\ldots\right]$$

$$= 2\left(1-\frac{x}{8}-\frac{x^2}{128}-\frac{x^3}{1024}+\ldots\right)$$ Multiply by 2

$$= 2-\frac{x}{4}-\frac{x^2}{64}-\frac{x^3}{512}+\ldots$$

(b) $(4-x)^{\frac{1}{2}}(1+2x)$ Use answer from part (a)

$= \left(2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \ldots\right)\left(1+2x\right)$ Multiply out brackets

$= 2 - \frac{x}{4} - \frac{x^2}{64} - \frac{x^3}{512} + \ldots + 4x - \frac{x^2}{2} - \frac{x^3}{32} + \ldots$ Collect like terms

$= 2 + \frac{15}{4}x - \frac{33}{64}x^2 - \frac{17}{512}x^3 + \ldots$

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 5

Question:

(a) Find the first four terms of the expansion, in ascending powers of x, of

$(2+3x)^{-1}, \ |x| < \frac{2}{3}$

(b) Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of:

$\frac{1+x}{2+3x}, \ |x| < \frac{2}{3}$ **E**

Solution:

(a) $(2+3x)^{-1}$ Take out factor of 2

$= \left[2\left(1+\frac{3x}{2}\right)\right]^{-1}$

$= 2^{-1}\left(1+\frac{3x}{2}\right)^{-1}$ Use binomial expansion with $n = -1$ and $x = \frac{3x}{2}$

$= \frac{1}{2}\left[1+\left(-1\right)\left(\frac{3x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{3x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{3x}{2}\right)^3+\ldots\right]$

$= \frac{1}{2}\left(1-\frac{3}{2}x+\frac{9}{4}x^2-\frac{27}{8}x^3+\ldots\right)$ Multiply by $\frac{1}{2}$

$= \frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3+\ldots$

Valid for $\left|\frac{3x}{2}\right| < 1 \Rightarrow |x| < \frac{2}{3}$

(b) $\frac{1+x}{2+3x}$ Put in index form

$= (1+x)(2+3x)^{-1}$ Use expansion from part (a)

$$= \left(1 + x \right) \left(\frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \quad \ldots \right)$$ Multiply out

$$= \frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \frac{1}{2}x - \frac{3}{4}x^2 + \frac{9}{8}x^3 + \quad \ldots$$ Collect like terms

$$= \frac{1}{2} - \frac{1}{4}x + \frac{3}{8}x^2 - \frac{9}{16}x^3 + \quad \ldots$$

Valid for $\left| \frac{3x}{2} \right| < 1 \Rightarrow |x| < \frac{2}{3}$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 6

Question:

Find, in ascending powers of x up to and including the term in x^3, the series expansion of $(4+x)^{-\frac{1}{2}}$, giving your coefficients in their simplest form. **E**

Solution:

$$(4+x)^{-\frac{1}{2}} = \left[4\left(1+\frac{x}{4}\right)\right]^{-\frac{1}{2}}$$ Take out factor of 4

$$= 4^{-\frac{1}{2}}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}} \qquad 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2}$$

$$= \frac{1}{2}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$$ Use binomial expansion with $n = -\frac{1}{2}$ and $x = \frac{x}{4}$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!} + \ldots\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{8}x + \frac{3}{128}x^2 - \frac{5}{1024}x^3 + \ldots\right)$$

$$\simeq \frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2 - \frac{5}{2048}x^3$$

Valid for $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$

Solutionbank

Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 7

Question:

$f(x) = (1 + 3x)^{-1}, \ |x| < \frac{1}{3}$.

(a) Expand f(x) in ascending powers of x up to and including the term in x^3.

(b) Hence show that, for small x:

$\frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3$.

(c) Taking a suitable value for x, which should be stated, use the series expansion in part (b) to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places. (E)

Solution:

(a) $(1 + 3x)^{-1}$ Use binomial expansion with $n = -1$ and $x = 3x$

$= 1 + \left(-1\right)\left(3x\right) + \frac{(-1)(-2)(3x)^2}{2!} + \frac{(-1)(-2)(-3)(3x)^3}{3!} + \ldots$

$= 1 - 3x + 9x^2 - 27x^3 + \ldots$

(b) $\frac{1 + x}{1 + 3x} = \left(1 + x\right)(1 + 3x)^{-1}$ Use expansion from part (a)

$= (1 + x)(1 - 3x + 9x^2 - 27x^3 + \ldots)$ Multiply out

$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3 + \ldots$ Collect like terms

$= 1 - 2x + 6x^2 - 18x^3 + \ldots$ Ignore terms greater than x^3

Hence $\frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3$

(c) Substitute $x = 0.01$ into both sides of the above

$$\frac{1 + 0.01}{1 + 3 \times 0.01} - \quad 1 - 2 \times 0.01 + 6 \times 0.01^2 - 18 \times 0.01^3$$

$$\frac{1.01}{1.03} \simeq \quad 1 - 0.02 + 0.0006 - 0.000018 \,, \qquad \left[\frac{1.01}{1.03} = \frac{101}{103} \right]$$

$$\frac{101}{103} \simeq \quad 0.980582 \qquad \text{Round to 5 d.p.}$$

$$\frac{101}{103} \simeq \quad 0.98058 \qquad (5 \text{ d.p.})$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 8

Question:

Obtain the first four non-zero terms in the expansion, in ascending powers of x, of the function f(x) where $f(x) = \frac{1}{\sqrt{(1+3x^2)}}$, $3x^2 < 1$. (E)

Solution:

$$f(x) = \frac{1}{\sqrt{1+3x^2}} = \left(\sqrt{1+3x^2}\right)^{-1}$$

$$= (1+3x^2)^{-\frac{1}{2}}$$ Use binomial expansion with $n = -\frac{1}{2}$ and $x = 3x^2$

$$= 1 + \left(-\frac{1}{2}\right)(3x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(3x^2)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(3x^2)^3}{3!} + \ldots$$

$$\simeq 1 - \frac{3x^2}{2} + \frac{27x^4}{8} - \frac{135x^6}{16}$$

Valid for $|3x^2| < 1$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 9

Question:

Give the binomial expansion of $(1+x)^{\frac{1}{2}}$ up to and including the term in x^3.

By substituting $x = \frac{1}{4}$, find the fraction that is an approximation to $\sqrt{5}$.

Solution:

Using binomial expansion

$$(1+x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^3}{3!} + \ldots$$

$$\simeq 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

Expansion is valid if $|x| < 1$.

Substituting $x = \frac{1}{4}$ in both sides of expansion gives

$$\left(1 + \frac{1}{4}\right)^{\frac{1}{2}} \simeq 1 + \frac{1}{2} \times \frac{1}{4} - \frac{1}{8} \times \left(\frac{1}{4}\right)^2 + \frac{1}{16} \times \left(\frac{1}{4}\right)^3$$

$$\left(\frac{5}{4}\right)^{\frac{1}{2}} \simeq 1 + \frac{1}{8} - \frac{1}{128} + \frac{1}{1024} \qquad \left[\left(\frac{5}{4}\right)^{\frac{1}{2}} = \sqrt{\frac{5}{4}}\right]$$

$$\sqrt{\frac{5}{4}} \simeq \frac{1145}{1024} \qquad \left[\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}\right]$$

$$\frac{\sqrt{5}}{2} \simeq \frac{1145}{1024}$$ Multiply both sides by 2

$$\sqrt{5} \simeq \frac{1145}{512}$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 10

Question:

When $\left(1 + ax \right)^{n}$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 27 respectively.

(a) Find the values of a and n.

(b) Find the coefficient of x^3.

(c) State the values of x for which the expansion is valid. (E)

Solution:

(a) Using binomial expansion

$$\left(1 + ax \right)^{n} = 1 + n \left(ax \right) + \frac{n \left(n - 1 \right) \left(ax \right)^{2}}{2 !} + \frac{n \left(n - 1 \right) \left(n - 2 \right) \left(ax \right)^{3}}{3 !} + \ldots$$

If coefficient of x is -6 then $na = -6$ ①

If coefficient of x^2 is 27 then $\frac{n \left(n - 1 \right) a^{2}}{2} = 27$ ②

From ① $a = \frac{-6}{n}$. Substitute in ②:

$$\frac{n \left(n - 1 \right)}{2} \left(\frac{-6}{n} \right)^{2} = 27$$

$$\frac{n \left(n - 1 \right)}{2} \times \frac{36}{n^{2}} = 27$$

$$\frac{\left(n - 1 \right) 18}{n} = 27$$

$$\left(n - 1 \right) 18 = 27n$$

$$18n - 18 = 27n$$

$$-18 = 9n$$

$$n = -2$$

Substitute $n = -2$ back in ①: $-2a = -6 \Rightarrow a = 3$

(b) Coefficient of x^3 is

$$\frac{n(n-1)(n-2)a^3}{3!} = \frac{(-2)\times(-3)\times(-4)\times 3^3}{3\times 2\times 1} = -108$$

(c) $(1+3x)^{-2}$ is valid if $|3x| < 1 \Rightarrow |x| < \frac{1}{3}$

Solutionbank
Edexcel AS and A Level Modular Mathematics

The binomial expansion
Exercise D, Question 11

Question:

(a) Express $\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2}$ as a partial fraction.

(b) Hence or otherwise show that the expansion of $\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2}$ in ascending powers of x can be approximated to $5 - \frac{7x}{2} + Bx^2 + Cx^3$ where B and C are constants to be found.

(c) State the set of values of x for which this expansion is valid.

Solution:

(a) Let $\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} \equiv \frac{A}{(1+x)} + \frac{B}{(2+x)} + \frac{C}{(2+x)^2}$

$\Rightarrow \quad \frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} \equiv \frac{A(2+x)^2 + B(1+x)(2+x) + C(1+x)}{(1+x)(2+x)^2}$

Set the numerators equal:

$9x^2 + 26x + 20 \equiv A(2+x)^2 + B(1+x)(2+x) + C(1+x)$

Substitute $x = -2$: $\quad 36 - 52 + 20 = A \times 0 + B \times 0 + C \times (-1)$

$\Rightarrow \quad 4 = -1C$

$\Rightarrow \quad C = -4$

Substitute $x = -1$: $\quad 9 - 26 + 20 = A \times 1 + B \times 0 + C \times 0$

$\Rightarrow \quad 3 = 1A$

$\Rightarrow \quad A = 3$

Equate terms in x^2: $\quad 9 = A + B$

$\Rightarrow \quad 9 = 3 + B$

$\Rightarrow \quad B = 6$

Hence $\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} \equiv \frac{3}{(1+x)} + \frac{6}{(2+x)} - \frac{4}{(2+x)^2}$

(b) Using binomial expansion

$$\frac{3}{(1+x)} = 3(1+x)^{-1}$$
$$= 3\left[1 + (-1)(x) + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \ldots\right]$$
$$= 3(1 - x + x^2 - x^3 + \ldots)$$
$$= 3 - 3x + 3x^2 - 3x^3 + \ldots$$

$$\frac{6}{(2+x)} = 6(2+x)^{-1}$$
$$= 6\left[2\left(1 + \frac{x}{2}\right)\right]^{-1}$$
$$= 6 \times 2^{-1}\left(1 + \frac{x}{2}\right)^{-1}$$
$$= 6 \times \frac{1}{2}\left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$
$$= 3\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \ldots\right)$$
$$= 3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \ldots$$

$$\frac{4}{(2+x)^2} = 4(2+x)^{-2}$$
$$= 4\left[2\left(1 + \frac{x}{2}\right)\right]^{-2}$$
$$= 4 \times 2^{-2} \times \left(1 + \frac{x}{2}\right)^{-2}$$
$$= 4 \times \frac{1}{4} \times \left[1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3 + \ldots\right]$$
$$= 1 \times \left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \ldots\right)$$
$$= 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \ldots$$

Hence

$$\frac{9x^2 + 26x + 20}{(1+x)(2+x)^2} \equiv \frac{3}{(1+x)} + \frac{6}{(2+x)} - \frac{4}{(2+x)^2}$$
$$\simeq \left(3 - 3x + 3x^2 - 3x^3\right) + \left(3 - \frac{3x}{2} + \frac{3}{4}x^2 - \frac{3}{8}x^3\right) - \left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3\right)$$
$$\simeq 3 - 3x + 3x^2 - 3x^3 + 3 - \frac{3x}{2} + \frac{3}{4}x^2 - \frac{3}{8}x^3 - 1 + x - \frac{3}{4}x^2 + \frac{1}{2}x^3$$

– $5 - \frac{7x}{2} + 3x^2 - \frac{23}{8}x^3$

Hence $B = 3$ and $C = \frac{-23}{8}$

(c) $\frac{3}{(1 + x)}$ is valid if $|x| < 1$

$\frac{6}{(2 + x)}$ is valid if $|x| < 2$

$\frac{4}{(2 + x)^2}$ is valid if $|x| < 2$

Therefore, they *all* become valid if $|x| < 1$.