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bj

Inflation

I. Introduction

The recent claim of the BICEP2 collaboration of observation of CMB B-mode polarization is obviously of fundamental importance. While the BICEP2 claim has undergone criticism, and in any case requires confirmation, in what follows I will assume that the claim is correct.

One theme of this note is that it is not unreasonable to associate discrete-symmetry violation with inflationary physics and with the inflationary dark energy component in particular. This can lead to parity-violating, circularly-polarized graviton modes dominating the tensor-mode contribution to the CMB. My personal motivation was originally based on the notion that circularly polarized gravitons would of course lead to circular polarization of the observed CMB photons. This in turn would lead to the central importance of measuring the Stokes parameter V, which is sensitive to CMB circular polarization. However, this requires insertion of quarter-wave plates into the telescope optics. To do this is technically straightforward. But strategically it is not easy to commit setup and running time to a measurement which is, with high probability, likely to end up being a null measurement. Consequently, most collaborations have chosen not to try it. However, one collaboration has recently explored the territory, and has set the best limits (arXiv 1307.6090). Meanwhile, I eventually discovered that my own expectations were wrong: there is no easy mechanism which turns circularly polarized gravitons into circularly polarized CMB photons.

The second theme of this note has to do with this failure. While one part of the reason for my mistake was simple carelessness in scrutinizing the literature, another part was the difficulty I found in understanding the nitty-gritty details of inflationary cosmology. This may be due to nothing more than my advancing age, because I have available to me excellent pedagogical resources such as Baumann's TASI lectures, and Dodelson's textbook. The difficulty had to do with the plethora of details inhabiting the phenomenology----details that are the result of a half-century of splendid theoretical research, linked with an even more splendid sequence of accurate experiments. In the back of Dodelson's book there are a few pages which ease the pain. They contain a list of the symbols used in the many equation therein, explaining what they mean, and indicating on what page of the book they were first introduced. While the list is immensely useful, the fact remains that it contains 147 entries. It seemed to me that, even for my limited purposes, I encountered all 147.

So why was this note written? With regard to the first theme, I wanted to do my best to encourage experimentalists to measure V. The best way for a theorist like me to do that is to provide an interesting model which does exhibits circular polarization. This I did, and the first part of this note is a description of the model. It accounts for the inflationary dark energy itself in terms of a Lorentz-violating Fermion condensate containing both vector and axial charge. In order for the model to be simple and to succeed in accounting all by itself for all the inflationary dark energy, both vector and axial contributions are needed. Furthermore the formalism simplifies enormously when the condensate is pure chiral. The inflationary-graviton phenomenology is only dependent on one parameter, which is known in the loop-quantum-gravity community as the Barbero-Immirzi parameter γ . And the only requirement for the inflationary-graviton modes to have large circular polarization is that γ be smaller than unity.

The second reason for writing this note is that in grappling with the post-reheating inflation phenomenology, I came up with a different descriptive language. It is based on the observation that inflaton and graviton field-oscillator modes, when viewed in Hamiltonian language, become highly squeezed after horizon crossing. This implies that at reheating, their phase-space Wigner functions are extremely filamentary, extending out to very large values of canonical momentum. Nevertheless, because the Liouville theorem is operative, the area of the phase-space region where the Wigner function is nontrivial does not increase. As a consequence, the subsequent post-reheating evolution is not sensitive to the thickness of the filament.

But a thick filament can describe a single classical pulse of inflaton (or graviton) field, most conveniently expressed in terms of a quasi-monoenergetic coherent-state amplitude with a Gaussian pulse shape. In this way one can replace the real problem by a simpler problem, where during inflation one has a dilute gas of classical inflaton (graviton) pulses. After these degrees of freedom cross and recross the horizon, one is left with a dilute ensemble of temperature perturbations, each of which is localized in space (and in momentum space). At the time of recombination, this ensemble leaves imprints on the CMB sky, each of which is circular, with a radius of about 30 milliradians. And it certainly possible to choose the numbers in the model so that there are a large number of these little spots, but not so large a number that they overlap, The temperature profile from inflatons within these spots looks something like the logo of the Target big-box stores. However those from circularly polarized gravitons often have a twisted quadrupole pattern. And the bottom line is that, even though this model gives a totally different CMB sky than the one measured by WMAP and Planck, the resultant power spectra are identical. This has to be the case because in each case the primary input data were the squeezed-phase-space filaments at reheating.

In this note, this visualization tool has only been sketched out. But even these first-attempt rough sketches suffice to qualitatively describe not only the shape of the famous temperature power spectrum, but also the shapes of the five other polarization power spectra as well. And this is accomplished with hardly any equations at all. I look forward to seeing this idea pursued by a professional inflationeer---one who is not only proficient in managing the 147 parameters, but also proficient in computer graphics. As for myself, I may return to this subject in a future note.

Before going on, I must say that I still feel that measurement of the Stokes parameter V is an important frontier measurement. There do exist theoretical motivations, but of a different nature than expressed here (cf. e.g. arXiv 1307.6090 and references therein). I also must note that the visualization idea mentioned above has a prehistory (cf. e.g. arxiv 0202215). I learned of it via a note by Daniel Eisenstein, "The Acoustic Peak Primer", available from his home page.

In Section II, we review the basics of FRW cosmology, in order to establish notation and the general descriptive point of view.

In Section III, we introduce the formalism necessary to describe this Fermion-condensate model. It involves the first-order Einstein-Cartan Lagrangian, supplemented with the symmetry-violating interaction widely known as the Holst term. The coupling constant for this Holst term is the inverse of the aforementioned Barbero-Immirzi parameter, which is relevant if and only if there is a source of torsion in the description. This is the case for this model; the Fermion condensate serves as a source of torsion.

In Section IV, we apply this formalism to FRW cosmology. The condensate densities are a source of dark energy, and this is the most interesting consequence of the model. The axial condensate subtracts, and the vector condensate adds. All the dark energy is condensate if the vector and axial condensate densities have the same magnitude.

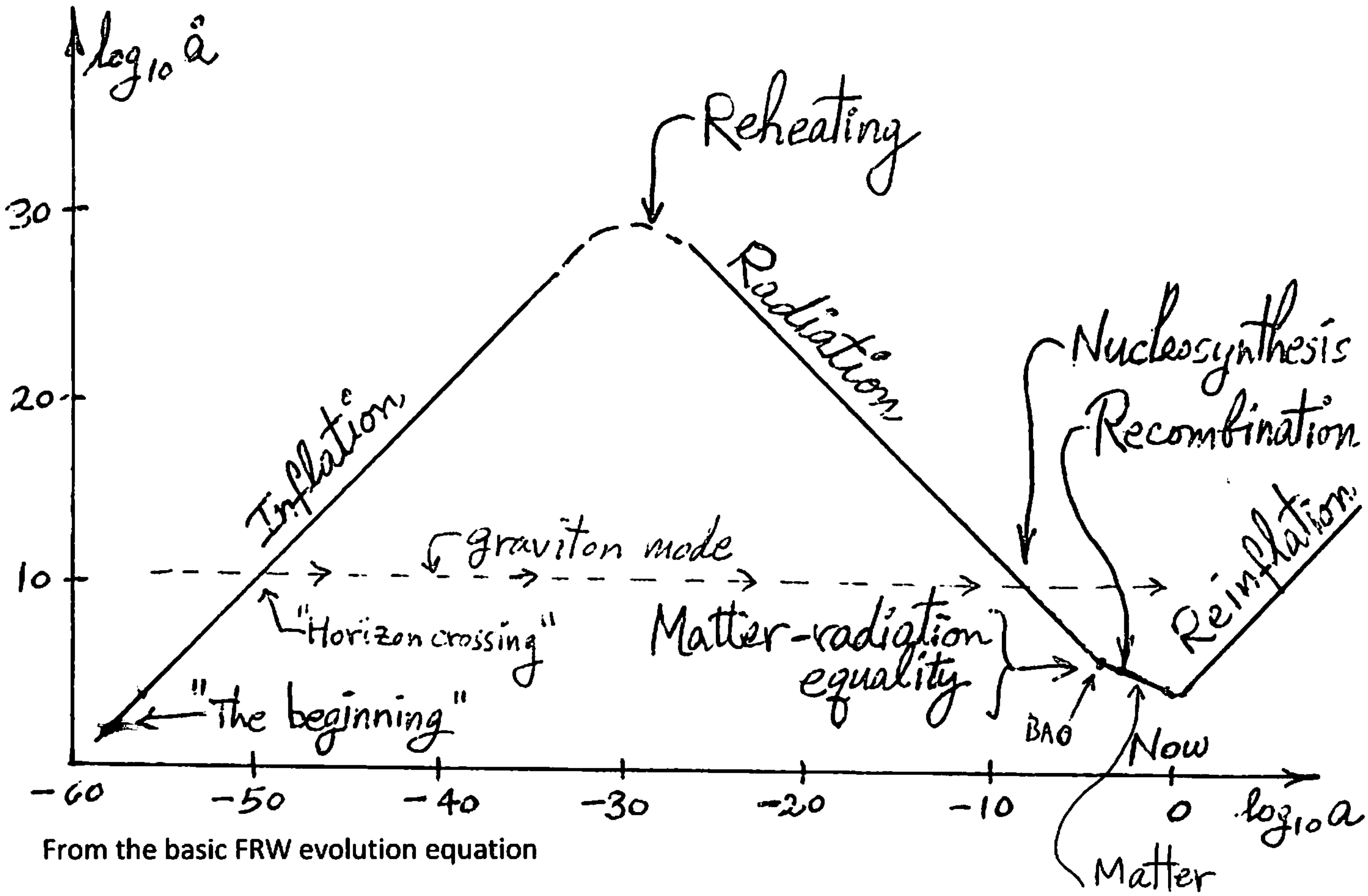
In Section V, we extend the Einstein-Cartan description to include transverse-traceless graviton degrees of freedom. Because the Einstein-Cartan description is simply an O(3,1) non-Abelian gauge theory, we expect and find that choice of temporal gauge leads to an especially user-friendly description. In particular, in the limit of a Minkowski-space (or DeSitter-space) background replacing the FRW background, the field equations have an exact gravitational plane-wave solution of the Einstein equations. But it is not a vacuum solution; the energy-momentum source term on the right hand side of the Einstein equation has properties consistent with what one expects from the energy-momentum carried by the gravitational wave itself. We then study the cosmological evolution of these transverse-traceless modes, all the way from early inflation to the present day, and exhibit the aforementioned squeezing of the modes after horizon crossing and in particular during the reheating period.

In Section VI, we study the evolution of the metric fluctuations after horizon recrossing, and how they leave an imprint on the CMB polarization power spectra. It is in this section that the visualization tools mentioned above are introduced.

Section VII contains a few concluding comments.

II. Cosmological Basics

In order to establish our descriptive language and make this note as self-contained as possible, we begin by reviewing the basics of cosmological history. This history is succinctly encoded in the behavior of the cosmological scale factor a(t) and of its time derivative $\dot{a}(t)$. This is plotted below, on a logarithmic scale:



From the basic FRW evolution equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3M_{pl}^2}\rho \qquad [\rho = \text{energy density}]$$

we see three distinct behaviors

Inflation:	$\rho = \text{const.}$	$a \sim e^{Ht}$	$\log \dot{a} = \log a + \text{const.}$
Radiation:	$\rho \sim a^{-4}$	$a \sim t^{1/2}$	$\log \dot{a} = -\log a + \text{const.}$
Matter:	$\rho \sim a^{-3}$	$a \sim t^{2/3}$	$\log \dot{a} = -\frac{1}{2}\log a + \text{const.}$

Also marked on the figure are landmark events:

1. Reheating: the end of inflation, when the inflationary dark energy is converted into predominantly extreme-relativistic quanta: "radiation". We have also included a period of time during which reheating proceeds and the universe continues to expand. It must be emphasized that everything to do with this event is speculative. There is not a scrap of data constraining the description of reheating.
2. Nucleosynthesis: From here forward to later times, there are considerable experimental constraints, while at earlier times there is much more freedom of theoretical choice.
3. Matter-radiation equality: This event will be very important in this note. After this event, the energy content of the universe was dominated by nonrelativistic quanta: "matter". This event is connected with the phenomenon of baryon acoustic oscillations (BAO), very relevant to the description of the CMB and polarization power spectra.
4. Recombination: This event marks the creation of neutral hydrogen from its ionized components. Thereafter the universe became transparent to photons. Hence the origin of the CMB photons can be traced back to this event but no further.
5. Nowadays: The importance of this event needs no elaboration.

We cannot resist commenting here on the absurdity of the descriptors "reheating" and "recombination". Prior to the reheating event, the default description of the very early universe is that it was always ice cold. And prior to recombination, there was never any time when neutral hydrogen atoms existed.

The horizontal line on the figure labeled "graviton" will become relevant as we trace the history of gravitational-wave degrees of freedom from their "birth" during inflation until the present time. It will be discussed in detail at the end of this section.

There is a second descriptive language which we will need. We will call it the conformal description, and is based on replacing the FRW time variable t with a new variable η .

The FRW line element is

$$ds^2 = dt^2 - a^2(t)\,dx^2$$

The new time variable η is defined as follows

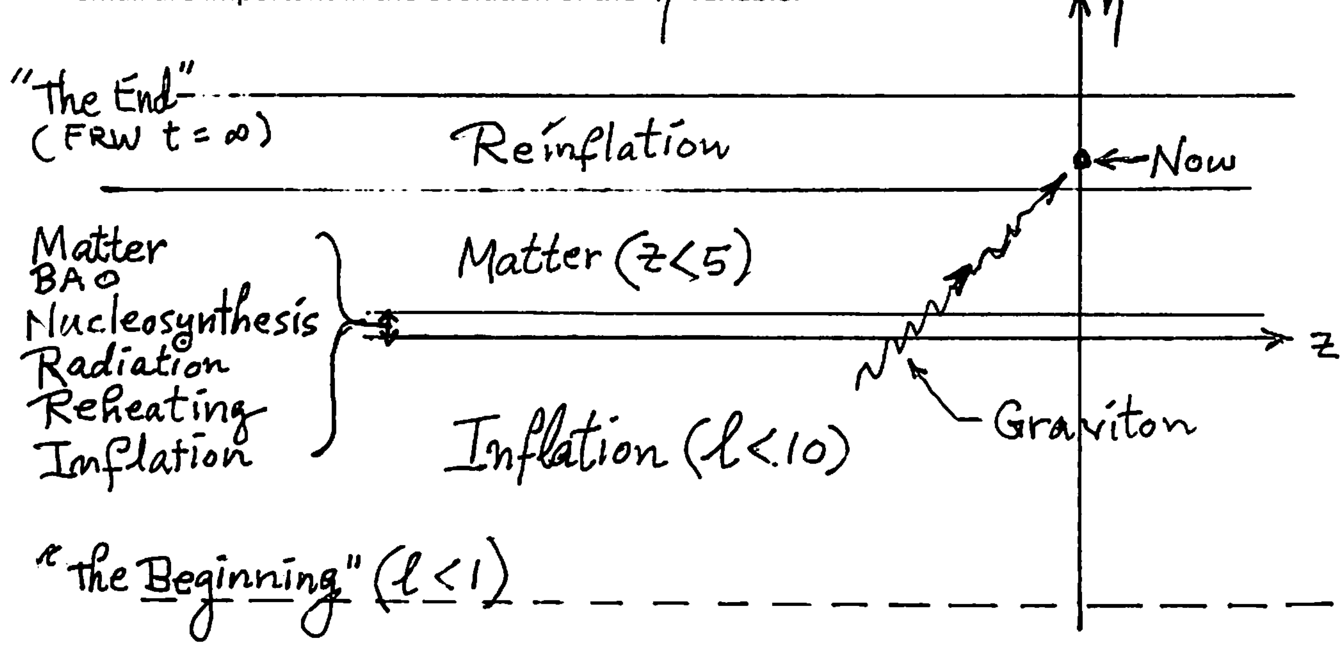
$$dt = a(t)\,d\eta \qquad\qquad \eta = \int_{-\infty}^{t} \frac{dt'}{a(t')}$$

Therefore the new metric is conformal

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\vec{x}\right)^2$$

This means that in this description the trajectories of photons and gravitons are straight lines, just as in Minkowski space. Therefore causal relationships are clearly revealed.

A plot of η versus a space coordinate z reveals that most of the cosmological history of the first figure is highly squished. Only a finite interval of η is needed to describe the entire history of the universe relevant to phenomenology. And only those regions for which $\dot{a}$ is very small are important in the evolution of the η variable:



We have drawn in this figure the trajectory of a graviton from its "birth" during inflation to its arrival to us at the present time. The history of this graviton is interesting. At birth its dynamics is essentially that of a plane wave. For a given wavelength, its dynamics is that of a harmonic oscillator. However, as the universe expands the wavelength will grow because the size of the box in which the plane wave is contained is expanding in proportion to the scale factor a , while the number of wavelengths within the box does not change. Therefore the frequency of the oscillator decreases in inverse proportion to a . These changes in the dynamics are well-accounted for by the adiabatic approximation until the period of the oscillator becomes comparable to the doubling time of the inflationary, expanding universe. Thereafter the oscillatory behavior is not possible, and the phase space description of the oscillator becomes that of a squeezed state. For irrational reasons, this event is called "crossing of the horizon".

The oscillatory behavior of the gravitational-wave mode does revive, however, during the hot big bang phase, when the expansion velocity $\dot{a}$ becomes sufficiently small. This second event is called "recrossing of the horizon". In general terms, the condition for crossing or recrossing the horizon is

$$|\omega| \sim \left|\frac{p}{a}\right| \sim \left|\left(\frac{\dot{a}}{a}\right)\right| \quad \Rightarrow \quad p \sim \dot{a}$$

Here p is the time-independent comoving wavenumber (momentum) of the mode. Therefore, for a given p , the history of the wave is encapsulated in the straight line drawn in the first figure. Low comoving momenta are near the bottom of the figure, while large comoving momenta are further up.

In inflation theory, these comoving momenta become proportional to the Legendre-function variables ℓ which parametrize the famous power spectrum. It is no accident that the "first acoustic peak" in that power spectrum occurs for a value of ℓ (and corresponding p) for which the horizontal line above intersects the BAO matter-radiation-equality point in Figure 1.

III. Lagrangians

As discussed in the introduction, we will describe the cosmological history prior to reheating in terms of a dark-energy-dominated deSitter space. The source of the dark energy will be assumed to be due to a fermionic, Lorentz-violating condensate, which also exhibits parity violation. This condensate will be assumed to be destroyed by the reheating event, leaving behind the hot big bang plus the tiny dark-energy residue which we see nowadays.

We therefore need to consider two distinct Lagrangians. They will differ by the nature of their source terms. But, in addition, the gravitational Lagrangian in the absence of sources that we will use will perhaps be unfamiliar. We use the first-order Einstein-Cartan formalism, supplemented by a CP-violating additional term, generally known as the Holst term. This formalism allows the information about the fermionic source to communicate to the gravitational sector via torsion degrees of freedom.

So we begin by writing down the gravitational Lagrangian. It describes a non-abelian gauge theory like QCD. But the gauge group is not SU(3)----it is the Lorentz group O(3,1). Instead of the 4 x 8 degrees of freedom describing the potentials of the eight gluons of QCD, we now have 4 x 6 gauge potentials to deal with. We actually will choose temporal gauge for everything we do, so that the number reduces to a mere 18, labeled as follows:

$$\omega_\mu^{AB} = \begin{array}{c} 01 \\ 02 \\ 03 \\ 23 \\ 31 \\ 12 \end{array} \begin{pmatrix} 0 & k_x^1 & k_y^1 & k_z^1 \\ 0 & k_x^2 & k_y^2 & k_z^2 \\ 0 & k_x^3 & k_y^3 & k_z^3 \\ 0 & c_x^1 & c_y^1 & c_z^1 \\ 0 & c_x^2 & c_y^2 & c_z^2 \\ 0 & c_x^3 & c_y^3 & c_z^3 \end{pmatrix} \qquad \omega_t^{AB} = 0$$

However, this is not the end of the story. The formalism also admits 4 x 4 = 16 more "vierbein" degrees of freedom, which provide the connection to the usual Einstein-Hilbert description based on the 10 degrees of freedom described by the metric tensor $g_{\mu\nu}$:

$$e^A_\mu = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & e^1_x & e^1_y & e^1_z \\ 0 & e^2_x & e^2_y & e^2_z \\ 0 & e^3_x & e^3_y & e^3_z \end{pmatrix} \qquad \begin{array}{l} g_{\mu\nu} = e^A_\mu e^B_\nu \eta_{AB} \\ \\ \eta_{AB} = \mathrm{diag}(1,-1,-1,-1) \end{array}$$

Again, for everything in this note, we can <u>ab</u> <u>initio</u> set 6 of these vierbein degrees of freedom to zero without getting into trouble. This has already been done in the above expression.

With these preliminaries, we can write down the basic gravitational Lagrangian in terms of these variables:

$$\mathcal{L}_{grav} \propto \epsilon^{\mu\nu\lambda\sigma} R^{AB}_{\mu\nu} e^C_\lambda e^D_\sigma \left[\epsilon_{ABCD} - \frac{2}{\gamma} \eta_{AC} \eta_{BD} \right]$$

The first factor is the standard Einstein-Cartan Lagrangian. The second factor is the Holst term. Its coupling constant γ is known as the Barbero-Immirzi parameter.

When we expand this Lagrangian out in terms of the temporal-gauge degrees of freedom, we find

$$\frac{8\pi}{3M_{pl}^2} \mathcal{L}_{grav} = \left[|e e \dot{k}| - N|ecc| + N|ekk| + N|(\partial e)c| \right] + \frac{1}{\gamma}\left[|e e \dot{c}| + 2N|ekc| - N|(\partial e)k| \right]$$

The individual terms are "double forms", which are invariant not only with respect to spatial rotations, but also with respect to O(3) gauge transformations. These "double forms" are defined as follows:

$$|abc| = \frac{1}{6} \sum_{\substack{ijk=1 \\ ABC=1}}^{3} \epsilon_{ABC}\, \epsilon^{ijk}\, a^A_i b^B_j c^C_k$$

$$|(\partial a)b| = \frac{1}{3} \sum_{\substack{ijk=1 \\ A=1}}^{3} \epsilon^{ijk} (\partial_i a^A_j)\, b^A_k \equiv |b(\partial a)|$$

This construction satisfies the following symmetry properties:

$$|abc| = |bac| = |bca|$$

The only independent degrees of freedom which possess time derivatives in the Lagrangian are

$$\tilde{k} = k + \frac{c}{\gamma}$$

This combination, well-known in the loop-quantum-gravity community, will evidently play a special role, and we therefore rewrite the Lagrangian appropriately:

$$\frac{8\pi}{3M_{pl}^2}\mathcal{L}_{grav} = |ee\dot{\tilde{k}}| - N\left(\frac{1+\gamma^2}{\gamma^2}\right)|ecc| + N|e\tilde{k}\tilde{k}|$$
$$- \frac{N}{\gamma}|(\partial e)\tilde{k}| + N\left(\frac{1+\gamma^2}{\gamma^2}\right)|(\partial e)c|$$

This gravitational Lagrangian still needs to be supplemented by source terms. After reheating, we only need the source term to describe the FRW expansion. A general function of the comoving volume $|e\,e\,e|$ suffices for this purpose. (In the usual Einstein-Hilbert metric language, the corresponding statement is that the source term need only depend upon $\sqrt{-||g||}$.)

$$\mathcal{L}_{FRW} = -N|eee|\,\rho(|eee|)$$

The normalization above is such that ρ is equal to the energy density which drives the expansion of the universe via the FRW equations of motion.

Before reheating, we are assuming that the inflationary dark energy is provided by a condensate of Dirac fermions. The appropriate Dirac Lagrangian is

$$\mathcal{L}_{Dirac} = \frac{i}{2}(1-i\alpha)\overline{\Psi}\gamma^A e_A^{\mu}\left(\partial_\mu + \frac{\omega_\mu^{BC}\gamma_B\gamma_C}{2}\right)\psi + h.c.$$

We have included a phase factor $(1 - i\alpha)$ out in front. In the Minkowski-space limit it does not affect anything. But in curved spacetime, as pointed out by Freidel et.al. (arXiv0507253), it does have an effect. (We have normalized this factor so that the Dirac action reduces to standard form in the Minkowski-space limit.)

We assume that these fermions form a condensate, such that

$$\rho_A = \langle \bar{\Psi} \gamma_5 \gamma_0 \Psi \rangle \neq 0 \qquad\qquad \rho_V = \langle \bar{\Psi} \gamma_0 \Psi \rangle \neq 0$$

In addition, we assume that the condensate is maximally parity-violating:

$$\rho_A = \rho_V$$

The Dirac Lagrangian describing the condensate reduces to

$$\langle \mathcal{L}_{Dirac} \rangle = -3N\left[\alpha \,|ee\tilde{k}|\, \rho_V + |eec|\, \rho_A\right]$$

This form demonstrates the role of the condensate---it provides a "$\vec{j}\cdot\vec{\omega}$" source term for the gravitational degrees of freedom---more precisely the FRW degrees of freedom.

It should not be too surprising to expect that additional simplicity will emerge if we choose

$$\alpha = \gamma \qquad\qquad \langle \mathcal{L}_{Dirac} \rangle = -3N\rho_A \gamma \,|ee\tilde{k}|$$

In what follows we will indeed make this choice. The more general case, however, has been considered, and some details can be found in another of these notes ("Gravitational Waves in the First Order Formalism").

IV. FRW Equations

For the FRW application, the double forms simplify enormously, because all the entries are diagonal:

$$\tilde{k} = \tilde{K}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad c = C\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad e = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|e\tilde{k}c| = a\tilde{K}C\text{, etc.}$$

It also follows immediately that, for this application,

$$|(\partial e)\tilde{k}| = |(\partial e)c| = 0$$

The inflationary Lagrangian (prior to the reheating event) therefore can be written

$$\frac{8\pi}{3M_{pl}^2}\mathcal{L}_{Inflation} = a^2 K - N\left(\frac{1+\gamma^2}{\gamma^2}\right)aC^2 + aN\tilde{K}^2 - \frac{8\pi}{M_{pl}^2}N\rho_A\gamma\left(a^2\tilde{K}\right)$$

The equations of motion are best obtained by varying C , $\tilde{K}$, and N in that order:

$$\delta C: \qquad C = 0$$

$$\delta \tilde{K}: \qquad 2a\dot{a} = 2aN\tilde{K} - \frac{8\pi}{M_{pl}^2} N\rho_A \gamma a^2$$

$$\delta N: \qquad a\tilde{K}^2 - \frac{8\pi}{M_{pl}^2}\rho_A \gamma a^2 \tilde{K} = 0$$

Once the variation with respect to N has been performed, we can set it equal to unity. While this is appropriate for the FRW description, it is not for the conformal description we will use for gravitational waves. In that case, to be discussed in the next section, we set N equal to $a(\eta)$.

The solution of these FRW equations can be immediately written down:

$$N = 1$$
$$C = 0$$
$$\tilde{K} = \left(\frac{8\pi}{M_{pl}^2}\gamma \rho_A\right) a$$

$$\left(\frac{\dot{a}}{a}\right) = \frac{\tilde{K}}{a} - \left(\frac{4\pi}{M_{pl}^2}\rho_A \gamma\right) = \frac{4\pi}{M_{pl}^2}\rho_A \gamma \equiv H \qquad \therefore a(t) = a(0)e^{Ht}$$

Note that we only get the deSitter-space solution if the Barbero-Immirzi parameter is positive. However, for negative γ , all we need to do is assume that $\rho_A = -\rho_V$. With ρ_V and α chosen positive, we then recover our simple solution. Hereafter, we assume that $\gamma > 0$.

Variation with respect to the "dreibein" variable leads to no new information, but provides a useful identity:

$$\delta a: \qquad 0 = 2a\dot{\tilde{K}} + \tilde{K}^2 - \frac{16\pi}{M_{pl}^2}\rho_A \gamma (a\tilde{K})$$
$$= 4H^2a^2 + 4H^2a^2 - 8H^2a^2 = 0$$

Before leaving this section, we also record the solution of the FRW equations for the epoch following reheating. In that case we need the general source term ρ . The Lagrangian now reads

$$\frac{8\pi}{3M_{pl}^2}\mathcal{L}_{FRW} = a^2\dot{\tilde{K}} - N\left(\frac{1+\gamma^2}{\gamma^2}\right)aC^2 + aN\tilde{K}^2 - \frac{8\pi}{3M_{pl}^2}Na^3\rho(a^3)$$

Variation with respect to C , $\tilde{K}$, and N lead to the FRW equations:

$$\delta C: \qquad C = 0$$

$$\delta K: \qquad 2a\dot{a} = 2aN\tilde{K}$$

$$\delta N: \qquad 0 = a\tilde{K}^2 - \frac{8\pi}{3M_{pl}^2}a^3\rho$$

The solution is, for the FRW metric,

$$N = 1$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\tilde{K}}{a}\right)^2 = \frac{8\pi}{3M_{pl}^2}\rho$$

Again, variation with respect to a leads to nothing more than another interesting identity;

$$\delta a: \quad 0 = 2a\dot{\tilde{K}} + \tilde{K}^2 - \frac{8\pi}{M_{pl}^2}a^2(\rho + a^3\rho')$$
$$= 2a\ddot{a} + \dot{a}^2 - \frac{8\pi}{M_{pl}^2}a^2(\rho + a^3\rho') = \frac{d}{dt}\left[\dot{a}^2 a - \frac{8\pi}{3M_{pl}^2}a^3\rho\right] = 0$$

V. Gravitational Waves

The description of gravitational waves is much simpler if the conformal metric is chosen, because the propagation of the waves becomes as simple as in Minkowski space. Therefore we record the modifications of the field equations which are a consequence of setting N equal to the scale factor a (η). Of course, in what follows, we can set the FRW contorsion C to zero everywhere. And our notation will be

$$\dot{a} = \frac{da}{d\eta}, \text{ etc.}$$

Before reheating, the field equations are

$$\delta N: \qquad \tilde{K} = \left(\frac{8\pi}{M_{pl}^2}\rho_A\gamma\right)a = 2Ha$$

$$\delta\tilde{K}: \qquad \frac{\dot{a}}{a^2} = \frac{\tilde{K}}{a} - H = H \qquad\qquad H = \frac{4\pi}{M_{pl}^2}\rho_A\gamma$$

Therefore

$$N = a(\eta)$$

$$a = -\frac{1}{H\eta}$$

The pre-reheating δa – identity becomes

$$\delta a: \qquad 0 = 2a\dot{\tilde{K}} + a\tilde{K}^2 - \frac{16\pi}{M_{pl}^2} a^2 \rho_A \gamma \tilde{K}$$
$$= 4H^2a^3 + 4H^2a^3 - 8H^2a^3 = 0$$

After reheating, the field equations become

$$N = a(\eta)$$
$$\delta\tilde{K}: \qquad \tilde{K} = \left(\frac{\dot{a}}{a}\right)$$
$$\delta N: \qquad \tilde{K}^2 = \frac{8\pi}{3M_{pl}^2} a^2 \rho$$

And the post-reheating δa – identity now becomes

$$\delta a: \qquad 0 = 2a\dot{\tilde{K}} + a\tilde{K}^2 - \frac{8\pi}{M_{pl}^2} a^3(\rho + a^3\rho')$$
$$= \left(\frac{a}{\dot{a}}\right)\frac{d}{d\eta}\left[a\tilde{K}^2 - \frac{8\pi}{3M_{pl}^2} a^3\rho\right] = 0$$

With these preliminaries, we are ready to introduce the transverse-traceless gravitational-wave degrees of freedom into the formalism. The FRW degrees of freedom will be considered a background in which the gravitons propagate. Of course, this constitutes an extremely good approximation.

We need to generalize the structure of the basic degrees of freedom $\tilde{k}$, c, and e , and of the double-forms which appear in the Lagrangian. We write

$$\tilde{k} = \begin{pmatrix} (\tilde{K}+\tilde{k}_+) & 0 & 0 \\ 0 & (\tilde{K}-\tilde{k}_+) & 0 \\ 0 & 0 & \tilde{K} \end{pmatrix} \qquad c = \begin{pmatrix} c_+ & c_\times & 0 \\ c_\times & -c_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e = \begin{pmatrix} (a+\epsilon_+) & 0 & 0 \\ 0 & (a-\epsilon_+) & 0 \\ 0 & 0 & a \end{pmatrix}$$

Expansion of the double-form structures yield only terms quadratic in the transverse-traceless variables:

$$|abc| = ABC - \frac{1}{3}\left[a_+b_+C + b_+c_+A + c_+a_+B\right] - \frac{1}{3}\left[a_\times b_\times C + b_\times c_\times A + c_\times a_\times B\right]$$

$$|(\partial a)\,b| = \frac{2}{3}\left[a_+'b_\times - a_\times' b_+\right]$$

We now begin with the pre-reheating description of the gravitational waves in the inflationary deSitter background metric. The transverse-traceless Lagrangian contains quite a few terms:

$$\frac{8\pi}{3M_{pl}^2}\left[\mathcal{L}_{grav}+\mathcal{L}_{Dirac}\right] = \left[-\frac{1}{3}\left(\epsilon_+^2+\epsilon_\times^2\right)\dot{\tilde{K}} - \frac{2}{3}\left(\epsilon_+\dot{k}_+ + \epsilon_\times\dot{k}_\times\right)a\right]$$

$$-N\left(\frac{1+\gamma^2}{\gamma^2}\right)\left[-\frac{1}{3}\left(c_+^2+c_\times^2\right)a\right]$$

$$+N\left[-\frac{1}{3}\left(k_+^2+k_\times^2\right)a - \frac{2}{3}\tilde{K}\left(\epsilon_+ k_+ + \epsilon_\times k_\times\right)\right]$$

$$+\frac{N}{\gamma}\cdot\frac{2}{3}\left[\epsilon_\times' \tilde{k}_+ - \epsilon_+'\tilde{k}_\times\right] - \frac{2}{3}N\left(\frac{1+\gamma^2}{\gamma^2}\right)\left[\epsilon_\times' c_+ - \epsilon_+' c_\times\right]$$

$$-\frac{8\pi}{M_{pl}^2}N\rho_A\gamma\left[-\frac{1}{3}\left(\epsilon_+^2+\epsilon_\times^2\right)\tilde{K} - \frac{2}{3}\left(\epsilon_+\tilde{k}_+ + \epsilon_\times\tilde{k}_\times\right)a\right].$$

We can clean this up by eliminating the FRW variables N , $\tilde{K}$, a , and $\rho_A\gamma$ in terms of η and H . After some algebra, we find the following Lagrangian:

$$\frac{8\pi}{3M_{pl}^2}\left[\mathcal{L}_{grav}+\mathcal{L}_{Dirac}\right] = \frac{2}{3\eta^2}\left(\epsilon_+^2+\epsilon_\times^2\right) + \frac{2}{3H\eta}\left(\epsilon_+\dot{k}_+ + \epsilon_\times\dot{k}_\times\right)$$

$$-\frac{1}{3H^2\eta^2}\left(k_+^2+k_\times^2\right) - \frac{2}{3\gamma H\eta}\left(\epsilon_\times' k_+ - \epsilon_+' k_\times\right)$$

$$+\frac{1}{3H^2\eta^2}\left(\frac{1+\gamma^2}{\gamma^2}\right)\left(c_+^2+c_\times^2\right) + \frac{2}{3H\eta}\left(\frac{1+\gamma^2}{\gamma^2}\right)\left[\epsilon_\times' c_+ - \epsilon_+' c_\times\right]$$

The subsequent equations of motion are

$$\frac{c_+}{H} = -\epsilon_\times'\eta \qquad\qquad \frac{c_\times}{H} = \epsilon_+'\eta$$

$$\frac{k_+}{H} = \epsilon_+ - \eta\dot{\epsilon}_+ - \frac{\eta}{\gamma}\epsilon_\times' \qquad\qquad \frac{k_\times}{H} = \epsilon_\times - \eta\dot{\epsilon}_\times + \frac{\eta}{\gamma}\epsilon_+'$$

$$2H\epsilon_+ = -\eta\dot{k}_+ + \frac{\eta}{\gamma}k_\times' - \eta\left(\frac{1+\gamma^2}{\gamma^2}\right)c_\times'$$

$$2H\epsilon_\times = -\eta\dot{k}_\times - \frac{\eta}{\gamma}k_+' + \eta\left(\frac{1+\gamma^2}{\gamma^2}\right)c_+'$$

Since these equations are linear, we can go to complex wave-equation notation. We assume a plane-wave structure with momentum p in the positive z direction. Therefore we make the replacements

$$k_+' = ip\, k_+ \qquad c_+' = ip\, c_+ \qquad \epsilon_+' = ip\, \epsilon_+$$
$$k_x' = ip\, k_x \qquad c_x' = ip\, c_x \qquad \epsilon_x' = ip\, \epsilon_x$$

The equations are simplified by the choice of a circular-polarization basis, which is implemented by the following assumptions

$$k_x = \lambda k_+$$
$$c_x = \lambda c_+ \qquad \lambda^2 = -1$$
$$\epsilon_x = \lambda \epsilon_+$$

It is easily seen that the wave equations are consistent with this ansatz. After some more algebra, we are left with a single wave equation for the amplitude ϵ_+:

$$\ddot{\epsilon}_+ + p^2 \epsilon_+ = 2\left[\frac{1}{\eta^2} - \frac{ip\lambda}{\gamma\eta}\right]\epsilon_+$$

This equation of motion is actually that of a harmonic oscillator with time-dependent frequency. This frequency passes through zero at horizon crossing and thereafter becomes imaginary. To clean up notation, we write

$$\tau \equiv p\eta \qquad \lambda = \pm i$$

The equation of motion is, in this streamlined notation,

$$\frac{d^2\epsilon_+}{d\tau^2} = \left[\frac{2}{\tau^2} \pm \frac{2}{\gamma\tau} - 1\right]\epsilon_+ \equiv -\hat{\omega}^2(\tau)\,\epsilon_+$$

Note that under a space reflection, which interchanges left- and right-handed chiralities, the term proportional to γ^{-1} changes sign,

For $\gamma = \infty$, it is known as the Mukhanov-Sasaki equation (cf. Baumann TASI lectures and/or his website notes on inflation) and has an exact solution:

$$\epsilon_+ = e^{i\tau}\, e^{-ipz}\left[1 - \frac{i}{\tau}\right]$$

It is interesting that even if we generalize the properties of the Dirac source, allowing unequal vector and axial condensate densities, the structure of the wave equation does not change: there is the same dependence on the value of the Barbero-Immirzi parameter and nothing else is changed.

After reheating, the description of the gravitational waves is similar, but not identical. The Dirac source term is replaced by the matter source term, the relevant parts of which are shown below:

$$\langle \mathcal{L}_{Dirac} \rangle \Rightarrow \mathcal{L}_{FRW} = -N|eee|\,\rho(|eee|) \Rightarrow \cdots + Na(\epsilon_+^2 + \epsilon_\times^2)(\rho + a^3\rho')$$

After using the FRW equations of motion to streamline things, we have

$$\frac{8\pi}{3M_{pl}^2}\left[\mathcal{L}_{grav} + \mathcal{L}_{FRW}\right] = \frac{1}{3}(\epsilon_+^2 + \epsilon_\times^2)\left(\frac{\ddot{a}}{a}\right) - \frac{2a}{3}(\epsilon_+\dot{k}_+ + \epsilon_\times \dot{k}_\times) - \frac{2}{3}\dot{a}(\epsilon_+ k_+ + \epsilon_\times k_\times)$$
$$+ \frac{a^2}{3}\left(\frac{1+\gamma^2}{\gamma^2}\right)(c_+^2 + c_\times^2) - \frac{a^2}{3}(k_+^2 + k_\times^2)$$
$$+ \frac{2a}{3\gamma}\left[\epsilon_\times' k_+ - \epsilon_+' k_\times\right] - \frac{2a}{3}\left(\frac{1+\gamma^2}{\gamma^2}\right)\left[\epsilon_\times' c_+ - \epsilon_+' c_\times\right]$$

The subsequent equations of motion are now, in streamlined form, quite simple;

$$\delta c: \qquad a c_+ = ip\lambda\, \epsilon_+$$
$$\delta k: \qquad a k_+ = \dot{\epsilon}_+ + \frac{ip\lambda}{\gamma}\epsilon_+$$
$$\delta \epsilon: \qquad \ddot{\epsilon}_+ + p^2\epsilon_+ = \left(\frac{\ddot{a}}{a}\right)\epsilon_+$$

The gravitational-wave equation is just the "generalized Mukhanov-Sasaki equation". It can be easily solved for the two simple post-reheating scenarios relevant in practice. If radiation dominates the expansion,

$$a(t) \sim t^{1/2} \qquad \eta = \int_0^t \frac{dt'}{a(t')} \sim t^{1/2} \sim a \qquad \therefore\ \ddot{a} = \frac{d^2a}{d\eta^2} = 0$$

Therefore the propagation is that of a plane wave:

$$\epsilon_+ \sim e^{-ip(\eta - z)}$$

On the other hand, if matter dominates over radiation,

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3} \qquad \eta \sim \int_{t_0}^{t} \frac{dt'}{a(t')} \sim \frac{t_0}{a_0}\left(\frac{t}{t_0}\right)^{1/3} \sim \frac{t_0}{a_0}\left(\frac{a}{a_0}\right)^{1/2}$$

Therefore

$$a \sim \eta^2\left(\frac{t_0^2}{a_0}\right)$$

This leads to exactly the same wave equation as we had for deSitter space (with $\gamma = \infty$), and that equation had an exact solution:

$$\epsilon_+ \sim e^{-ip(\eta - z)}\left(1 - \frac{i}{p\eta}\right)$$

In short, the post-reheating evolution of modes, before they "recross the horizon" essentially undoes much of the phase-space squeezing which occurred before reheating. But there are many details to consider. One must properly match the solutions at reheating, i.e. at $\eta \cong 0$, and to discuss the "BAO divide". This is defined by a particular value P of the comoving momentum p. For $p \gg P$, the "horizon recrossing" occurs when the universe is radiation dominated, while for $p \ll P$, the "horizon recrossing" occurs when the universe is matter dominated. Evidently, when $p \sim P$, things will be more complicated---and more interesting. This will be discussed in the next section.

However, before we move on, we will examine the phase-space properties of ensembles of classical solutions. This has the advantage that the quantum-mechanical issues and other subtleties that arise in the description of inflation can be visualized at the same time.

We start with the exact deSitter solution above as an example. It can be written

$$\epsilon_+ = -i\,e^{-i\tau}\left(1 - \frac{i}{\tau}\right) \equiv Q_1 + iQ_2 \qquad \dot{\epsilon}_+ \equiv P_1 + iP_2$$

The pairs (Q_1, P_1) and (Q_2, P_2) are solutions of the Hamiltonian equations of motion. They will serve as a basic skeleton for the description of a more general phase space architecture.

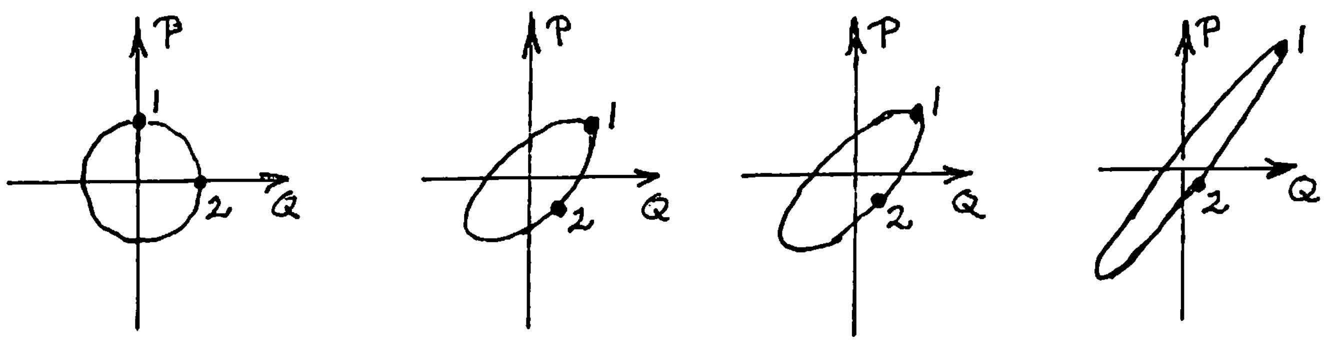
When τ is large (and negative!) we have a simple harmonic oscillator. Its natural phase-space structure is circular, both at the classical level and at the quantum level, provided a Bunch-Davies vacuum structure is assumed. We can create this circular structure by modifying our starting point by appending a phase factor on the complex amplitude ϵ_+:

$$\epsilon_+ \rightarrow \frac{e^{i\phi}}{\sqrt{2p}}\,\epsilon_+$$

We have also provided a canonical normalization for the state as well.

When τ becomes small, the mode "crosses the horizon" and no longer is oscillatory. The frequency parameter $-\tilde{\omega}^2$ in fact goes through zero and changes sign. Thereafter the phase space for the original solutions (with ϕ set to zero) is squeezed and rapidly turns into a filamentary structure. This can be seen by mapping the exact solutions as a function of τ :

$\tau = -\pi(2N+1)$, N large $\qquad \tau = -\frac{\pi}{2} \qquad \tau = -\frac{\pi}{3} \qquad \tau = -\frac{\pi}{6}$



Note that the pair (Q_2 , P_2) describes the thickness of the filament, while the pair (Q_1 , P_1) describes its length. The area of the filament is described by the quantity $Q_2P_1 - Q_1P_2$, independent of τ . This quantity, the Wronskian, is an expression of the Liouville theorem, which is applicable to this system.

For very small τ , the filament is almost vertical in the phase-space; the angle of deviation is $\sim \frac{1}{\tau}$. The appropriate variable to describe the coordinate is evidently

$$\left(\frac{\epsilon_+}{a}\right) \sim H\epsilon_+ \eta \sim \frac{H}{\sqrt{2p}}\left(\frac{1}{p\eta}\right)\eta \sim \frac{H}{p^{3/2}}$$

This remains finite in the limit.

A basic quantity which contains much of the inflationary source information which will eventually be expressed as experimental quantities is the correlation function of the fluctuations ("power spectrum"):

$$(2\pi)^3 \delta^3(p-p')\langle \epsilon_+(p,\eta)\, \epsilon_+^*(p',\eta)\rangle = \int d^3x\, e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')} \langle \epsilon_+(x,\eta)\, \epsilon_+^*(x',\eta)\rangle \sim \frac{H^2 a^2}{p^3}\delta^3(p-p')$$

Because different momentum modes are independent, this correlation function is diagonal in momentum space. In the next section, we will consider its post-reheating evolution as a function of conformal time τ. An especially important event will be at recombination, when the CMB photons that we observe nowadays were created. They are described by similar correlation functions, which in fact are linearly related to the above correlation function of the fluctuations, evaluated, as here, at an early enough conformal time so that the modes of interest had not yet recrossed the horizon.

VI. Observable Consequences: The Power Spectra

The metric fluctuations created during inflation are eventually expressed as a pattern on the sky of CMB photon temperature and polarization fluctuations. The polarization of photons arriving from, say, the z direction is described in terms of the 2 x 2 Hermitian Stokes matrix:

$$P_{ij} = \frac{1}{2}\left(I + \vec{\tau}\cdot\vec{S}\right) \qquad \begin{matrix} i=1 \Leftrightarrow x \\ i=2 \Leftrightarrow y \end{matrix}$$

Here the $\vec{\tau}$ are the Pauli matrices, and the coefficients, in a often-used notation which we adopt, are

$$S_3 \equiv Q$$
$$S_1 \equiv U$$
$$S_2 \equiv V$$

A similar description exists for gravitational waves. For a fully polarized beam, P_{ij} is a projection operator:

$$P^2 = P$$
$$\det P = 0$$

	Photon	Graviton
$\lvert Q\rvert = 1$	$\longleftrightarrow$	$+$
$\lvert U\rvert = 1$	$\updownarrow$	$\times$
$\lvert V\rvert = 1$	R, L	R, L

In the model of inflation that we have described, the parameter V(graviton) is large, and the other two are zero. While one might expect this information to be transmitted to the CMB Stokes parameter V(photon), the opposite is the case. The information does get transmitted, but it is only to the other Stokes parameters Q(photon) and U(photon). The way this happens is not especially simple to describe. But it was worked out in detail long ago and is well documented in the literature (cf. eg. arXiv 0705.3701 and arXiv 1002.1308).

I will not reproduce the details here. The main reason for this is that I am not competent to do so. Nevertheless, I will attempt a semiquantitative description. I find the discussions in the literature rather opaque and crave an intuitive understanding of what is going on. So what follows is an attempt to provide some simple understanding of the relevant physics.

The bottom-line quantities which emerge from the data are power spectra. The prime example is the famous power spectrum of the CMB temperature fluctuations. Define, for small θ(which is all that we will need),

$$C^{TT}(\theta) = \int \frac{d\Omega_n}{4\pi} \int \frac{d\phi}{2\pi}\, T\left(\hat{n} + \frac{\vec{\theta}}{2}\right) T\left(\hat{n} - \frac{\vec{\theta}}{2}\right)$$

In this expression, $\hat{n}$ is a unit vector pointing in some direction in the sky. Rotations around n are then performed, along with averaging $\hat{n}$ over the celestial sphere, while keeping the opening angle $|\vec{\theta}|$ constant. Then the resultant correlation function is expanded in Legendre functions. The coefficients define the CMB TT power spectrum:

$$C^{TT}(\theta) = \sum_{\ell} C_{\ell}^{TT}\, P_{\ell}(\cos\theta)$$

$$C_{\ell}^{TT} = \left(\ell + \frac{1}{2}\right) \int_{-1}^{1} d(\cos\theta)\, P_{\ell}(\cos\theta)\, C^{TT}(\theta)$$

In the same way, power spectra can be defined for the CMB-polarization Stokes parameters, leading to six distinct observables in all: TT, UU, VV, TU, TV, UV. These observables all emerge from the power spectrum for the inflationary fluctuations defined in the previous section. The problem is to connect the former with the latter. In the literature this is done by decomposing the 3-D temperature map into Fourier modes. These evolve with time independently (This is indeed the essential simplifying feature.) While the temperature map is simple at very early (conformal) times following reheating, the subsequent evolution is not. What we will do is to deal directly with the description on the sky at recombination and infer the power spectrum (approximately) without ever looking at the individual Fourier modes.

We found that the correlation function for scale-invariant, right-handed graviton degrees of freedom is very simple at very small η. This simplicity remains after the graviton degrees of freedom cross the horizon, because they do not interact with the hot Big Bang plasma. Complications do occur when the information contained in these gravitational degrees of freedom is communicated to the plasma and eventually to the CMB photons which are nowadays observed. This transfer of information involves somewhat subtle dynamical mechanisms, along with complications of a geometrical nature.

There will be five FRW times of importance in what follows, which lead to five distinct values of the conformal-time variable η :

1. Initiation of the hot big bang: all modes are inside the horizon ($\eta = \eta_R$)
2. Horizon crossing for a typical large-p mode under consideration ($p\eta_p = 1$)
3. Matter-radiation equality ($\eta = \eta_{eq}$; also $\eta_{eq} P = 1$)
4. Recombination ($\eta = \eta^*$)
5. Nowadays ($\eta = 1 + \eta^*$)

It will be important to keep track of the values of these times. Note that, because the conformal time scale is inversely proportional to the FRW scale factor, which itself has no absolute normalization scale for a spatially flat universe, we can normalize the conformal time interval between recombination and nowadays to unity. This is a convenient choice because it relates the CMB angular and angular-momentum variables directly to η and to the comoving momentum p .

The landmark values of conformal time are as follows:

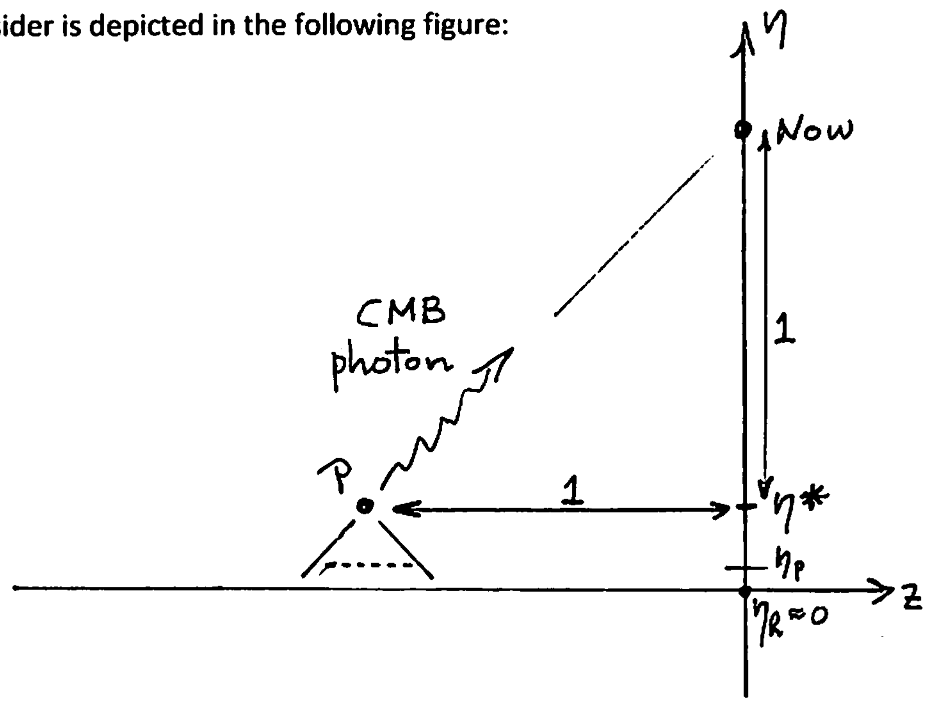
$$\frac{(1+\eta^*)}{\eta^*} \approx (1100)^{1/2} \sim 33 \iff \eta^* \approx .030$$

$$\frac{(1+\eta^*)}{\eta_{eq}} \approx (3700)^{1/2} \sim 60 \iff \eta_{eq} \approx .017$$

$$\eta_p = \eta_{eq} \quad (p > P) \iff \eta_p \approx .017\left(\frac{P}{p}\right) \quad (p > P)$$

$$\frac{\eta_{eq}}{\eta_R} \sim (10^{22})^{1/2} \sim 10^{11} \iff \eta_R \approx .000$$

The geometry we must consider is depicted in the following figure:

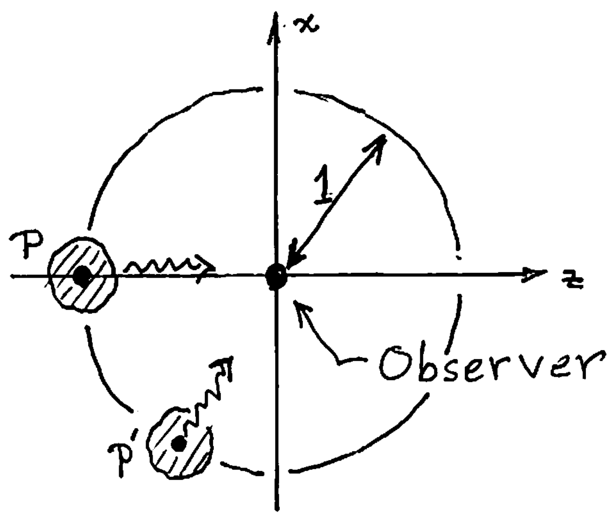


The CMB photons we observe were created at a redshift of (1100 ± 100) --- almost, but not quite, simultaneously. The prehistory of such photons is that they underwent Compton scattering (more precisely Thomson scattering) from an electron. While the incident photon in the Thomson-scattering process had the same energy as the final CMB photon, its direction differed. The Thomson cross section

$$\frac{d\sigma}{d\Omega} \sim |\vec{\epsilon} \cdot \vec{\epsilon}\,'|^2$$

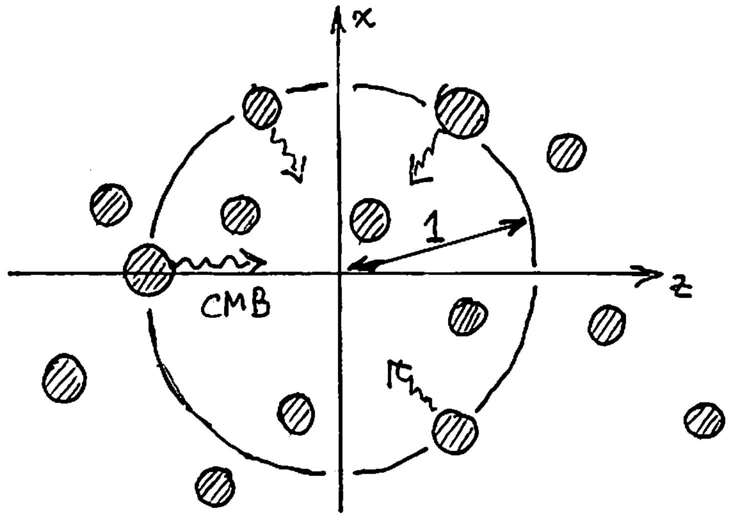
correlates incident and final photon polarizations in a nontrivial way. In principle, therefore, the polarization of the CMB photon can be dependent on the contents of the past lightcone of the point of emission P of the CMB photon --- at least within the conformal-time interval $\eta_R < \eta < \eta^*$.

But a plan view of the situation shows that there is only a short-range correlation between CMB photons coming from different directions in the sky.



We are now ready to introduce the central idea underlying the simplified description which will follow. It is based on the observation, made in the previous section, that the quantum state of a graviton mode after horizon crossing is highly squeezed. It is a thin, long filament in phase space. The subsequent evolution of the sundry fluctuations of photons, electrons, dark matter, and protons which are induced by the graviton mode after reheating are described by a plethora of Boltzmann equations. But these do not depend upon the thickness of the phase-space filament. Therefore we will replace the dense fluid of graviton degrees of freedom by a "dilute gas" of graviton modes, each of which is semiclassical (the mean number of gravitons in the mode is very large compared to unity) and which is described by a coherent state: a wave packet which is almost a plane wave but is modulated by a Gaussian envelope. If the phase space structure of this packet is, before horizon crossing, circular but with an area large compared to unity, then after horizon crossing it will be squeezed just like the quantum packet.

With this initial condition, the state at reheating can be taken to be a dilute gas of individual classical modes which are well separated in space. They will evolve after reheating independently. The situation at recombination is shown below:

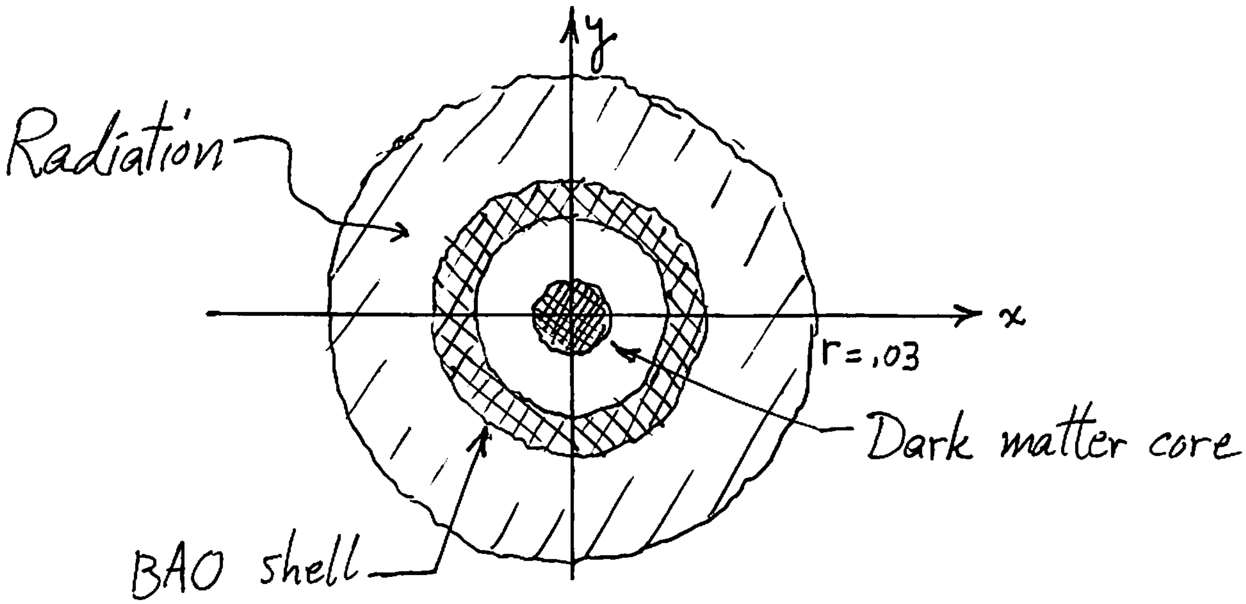


The individual fluctuations are now uncorrelated. Unlike the real situation, they do not overlap in space. Nevertheless, the power spectra will be the same, because they only depend on the phase-space evolution of the individual momentum-space modes. These in turn will be the same because of the filamentary structure of the individual modes at reheating. They contain all the initial-state information used by the Boltzmann-equation evolution.

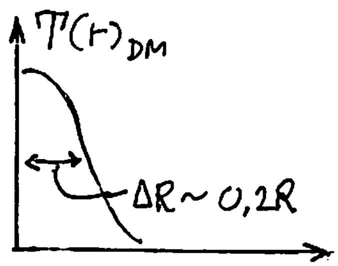
We now exhibit how this works by considering the temperature fluctuations induced by inflaton degrees of freedom. We consider a classical inflaton wave packet which crosses the horizon and creates a fluctuation which recrosses the horizon at $x = y = 0$, and at a value of z such that its CMB photons will be observed. We assume it has a very high momentum p , large compared to the characteristic momentum P associated with matter-radiation equality.

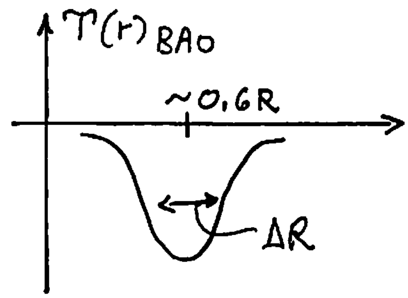
The packet reenters the horizon at a conformal time η_R much smaller than η^*. The inflaton degree of freedom, of course, disappeared during reheating. All that is left at horizon reentry is a hot spot of small size. As time goes on, this hot spot creates a disturbance to the relativistic plasma components. It travels outward at the speed of light. The dark matter, however, does not respond. But the protons and electrons do respond via the BAO (baryon acoustic oscillation) phenomenon, which plays a significant role. The BAO sound waves propagate at a sound velocity less than the speed of light, $c_s \approx c/\sqrt{3}$.

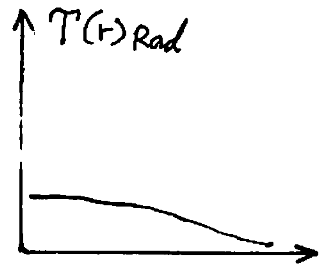
Therefore, by the time we reach the recombination era, the overdensity will be contained in a sphere of radius 0.03 or so, but with a substructure consisting of a dark-matter core at small radii surrounded by a halo of fluctuating density created by the outgoing BAO sound wave. In addition there may be a more diffuse component with the full radius of 0.3 consisting of the relics of the radiation-dominated era. These are schematically depicted in the figure below:



The power spectra used in the full analyses involve decomposition of the temperature fluctuations into spherical harmonics. But for our purposes here, this is overkill. Since we are dealing with a very small patch of sky, there is little need to use anything but Fourier transforms. Each of the density profiles above admits a simple Fourier transform:

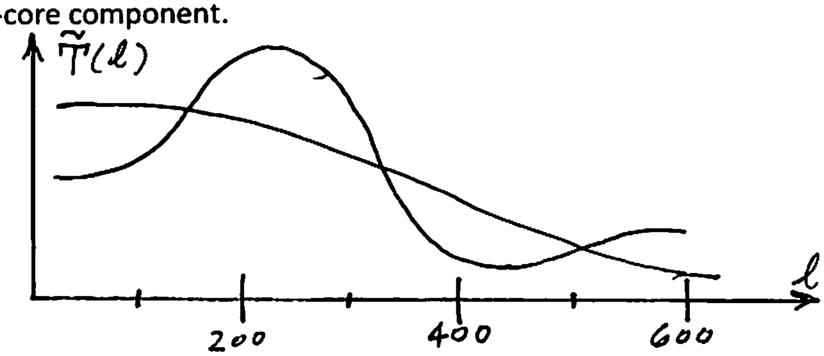






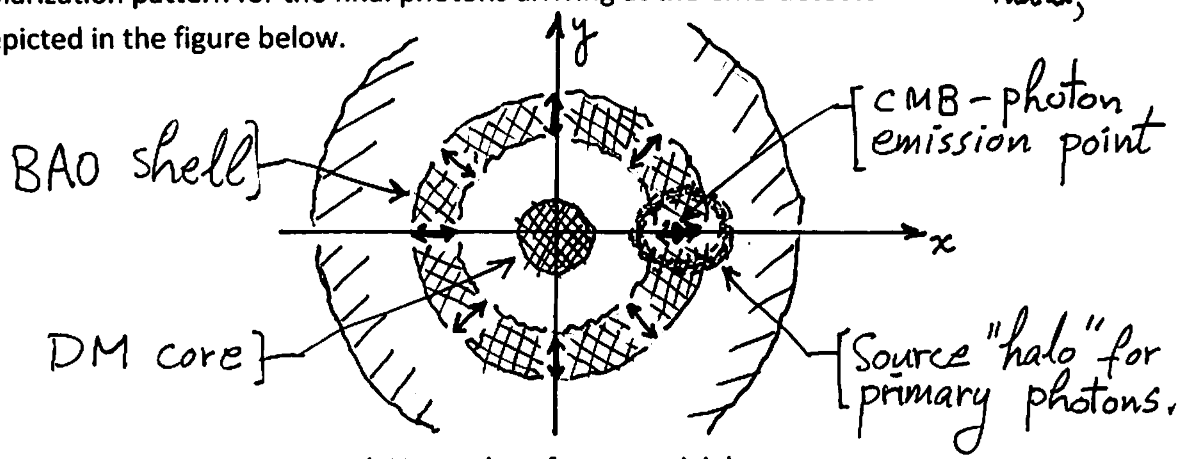
$$\tilde{T}(\ell)_{DM} \sim e^{-\frac{\ell^2(\Delta R)^2}{2}} \qquad \tilde{T}(\ell)_{BAO} \sim \frac{(\sin R_B \ell)}{\ell}\, e^{-\frac{\ell^2(\Delta R)^2}{2}} \qquad \tilde{T}(\ell)_{rad} \sim e^{-\frac{\ell^2 R^2}{2}}$$

We can easily get something that looks like the famous TT power spectrum by neglecting the diffuse component and subtracting the BAO halo component (which oscillates with time) from the dark-matter-core component.



The variable ℓ is properly labeled, given the conjugate relationship between angular momentum and angle when the ℓ's are large and the angles are small.

Our primary goal is not to discuss the TT power spectrum in detail. So we turn to the polarization power spectra. We expect the BAO-shell component to possess the induced polarization via the Thomson scattering. If the relevant photons originate nearby (cf. the figure below), they will propate inward to the scattering point and then undergo, on average, a 90 degree scatter into the final z-direction en route to us. The initial-photon polarization component in the z direction will not survive the Thomson scatter. Therefore the expected polarization pattern for the final photons arriving at the CMB detector will be radial, as depicted in the figure below.



The Stokes parameters Q and U are therefore nontrivial.

$$Q(r,\theta,\phi) \propto T(r)_{BAO} \cos 2\phi$$
$$U(r,\theta,\phi) \propto T(r)_{BAO} \sin 2\phi$$

At this point it is appropriate to introduce the E and B modes, which are in common use. They replace Q and U :

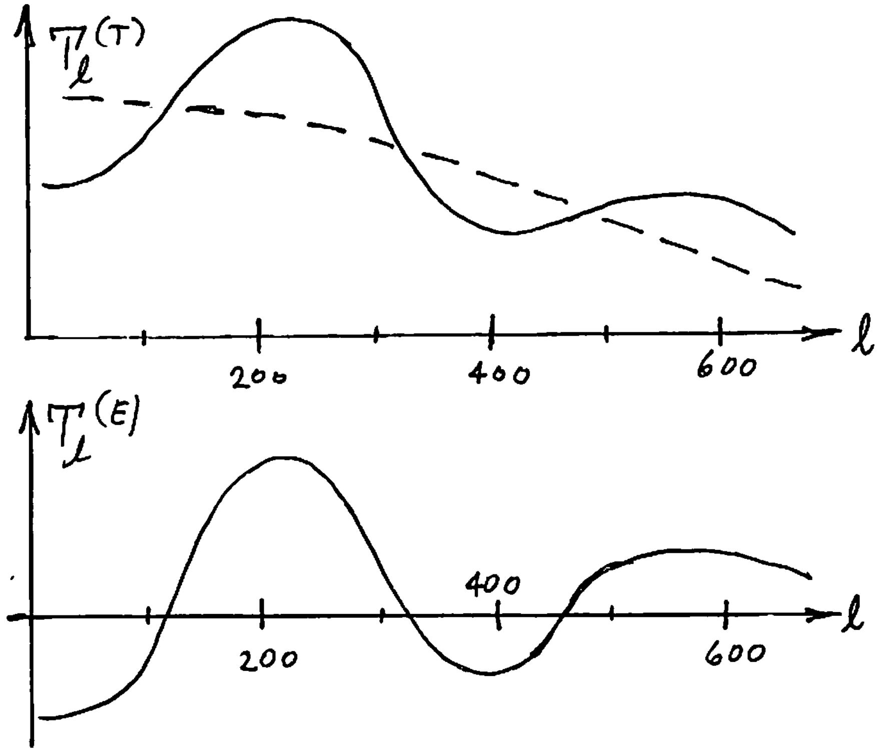
$$E = [\,Q \cos 2\phi + U \sin 2\phi\,] \propto T(r)_{BAO}$$
$$B = [-Q \sin 2\phi + U \cos 2\phi\,] = 0$$

The B mode vanishes whenever the polarization is totally radial or totally azimuthal. Azimuthal polarization could dominate if the important source of primary photons was in the DM core and not in the BAO shell.

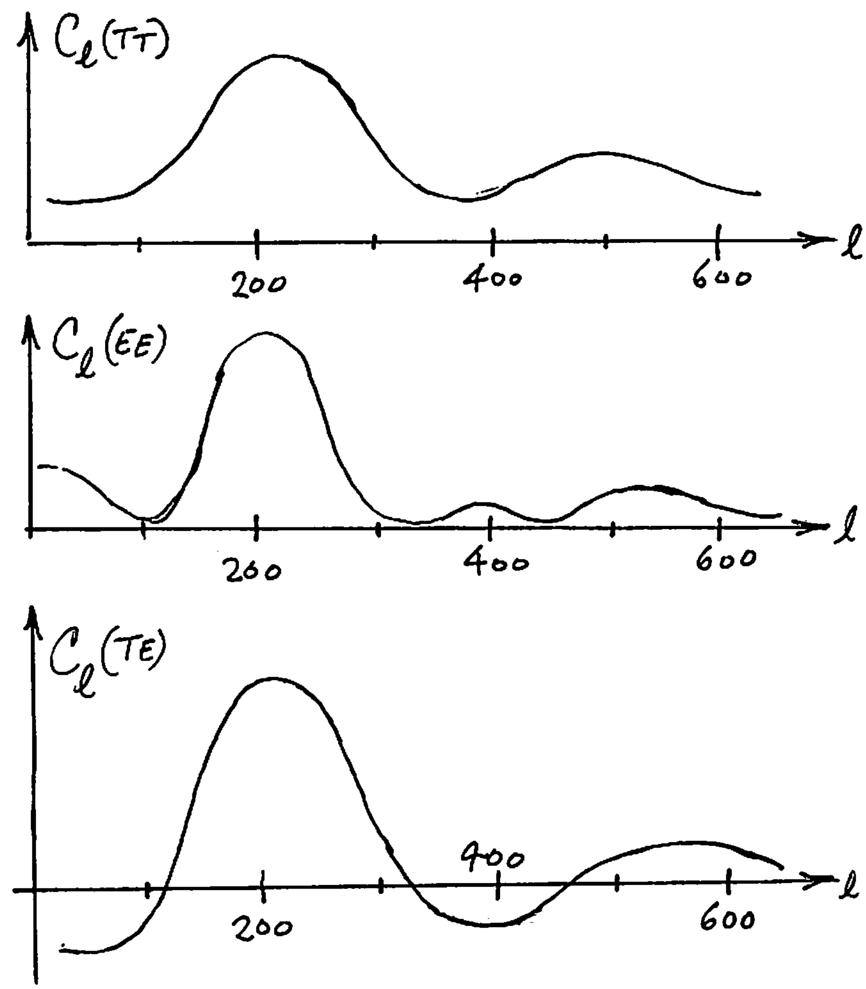
Because the E mode is azimuthally symmetric, it is relatively easy to process. In particular, from these expressions, we can directly construct the EE and TE correlation functions.

$$C_\ell^{TT} \sim |\tilde{T}_\ell^{DM} + \tilde{T}_\ell^{BAO}|^2$$
$$C_\ell^{EE} \sim |\tilde{T}_\ell^{BAO}|^2$$
$$C_\ell^{TE} \sim (\tilde{T}_\ell^{DM} + \tilde{T}_\ell^{BAO})\tilde{T}_\ell^{BAO}$$

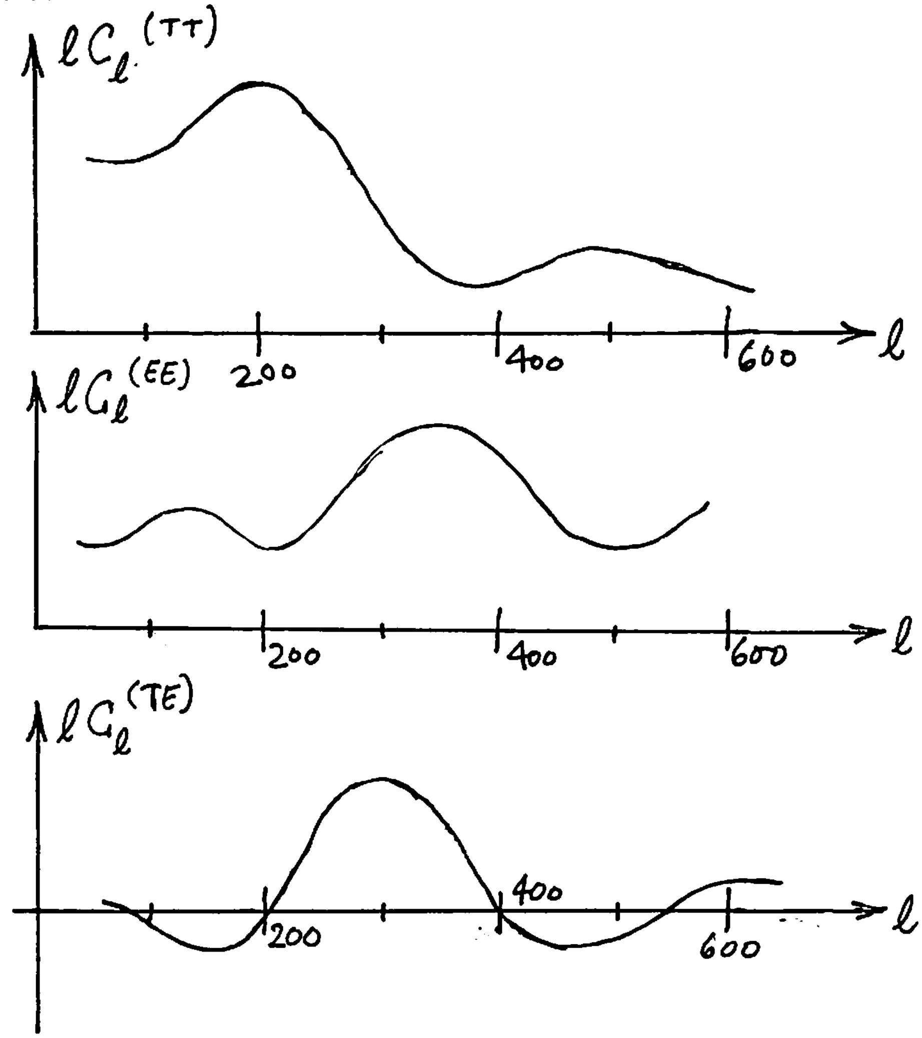
The T and E correlation functions are sketched below



These in turn allow sketches of the TT, EE, and TE power spectra:

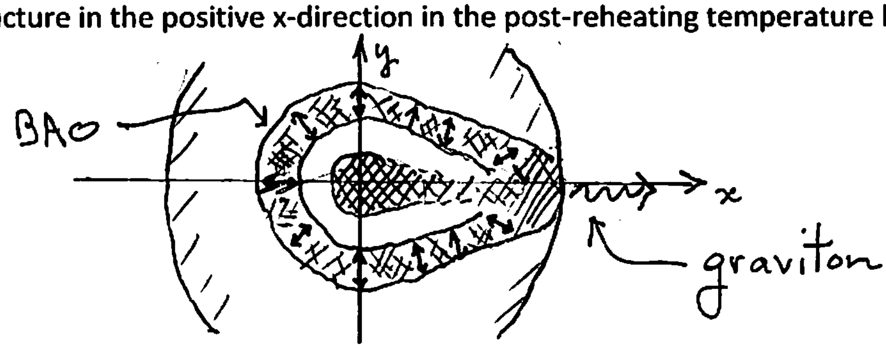


The power spectra from the literature (Dodelson, Baumann TASI lectures, arXiv 0705.3701) are plotted below:



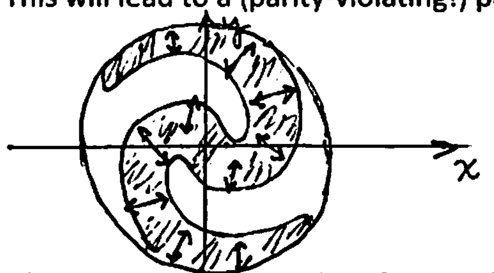
These are close enough to our sketches to provide encouragement that what we are doing, despite being very crude, is not total nonsense. Therefore we move on to inclusion of the tensor modes. (Note that there is a loose factor ℓ multiplying C which needs to be dealt with.)

The main difference from the case of the inflaton is that the graviton does not die during reheating, but continues into the post-reheating world. Consider the (quite typical!) case where the graviton propagates in a direction transverse to that of the CMB photon of interest. Without loss of generality, we can assume this to be the x-direction. After the graviton crosses the horizon, will disturb its local environment, and in particular will create a BAO shock wave. This leads to structure in the positive x-direction in the post-reheating temperature bubble:



The new feature is the loss of azimuthal symmetry, with structure at large x and small y . This leads to nonvanishing B polarization.

However, there is a less complex configuration as well. When the graviton is aimed at the observer, there will be a quadrupole structure already emergent at horizon crossing, but which rotates with conformal time. This will lead to a (parity-violating!) pattern as shown below:



Even averaging over azimuth, this pattern contains a lot of B- mode polarization. However, only a relatively small fraction of the gravitons (10-20 %) are on average aimed near enough to the z direction to create this strong pattern. However, we will assume that it is this pattern that dominates the B-mode contribution.

The distribution of B-mode polarization across the disc is broader than that of the E-mode polarization. Therefore we expect the BB, TB, and EB correlation functions to peak at a lower value of angular momentum ℓ. This corresponds to what is computed in detail and presented in the literature (cf. arXiv 1002.1308 and arXiv 0705.3701) ; the peaks of those power spectra occur for $\ell < 100$.

VII. Concluding Remarks

These notes contain two messages, each of which needs further elaboration:

1.) Symmetry violation in the inflationary era, especially with respect to the tensor-mode, gravitational-wave component, deserves to be taken very seriously at the level of phenomenology. This is an old idea; the contribution here is addition of yet another theoretical scenario of how this could happen. This scenario accounts for the inflationary dark energy, thanks to presence of a Lorentz-violating Fermion condensate.

2.) When I initiated this effort, I expected that circular polarization of inflationary gravitons would lead to circularly polarized CMB photons. This expectation is wrong, and was documented in detail long ago. Nevertheless, the imprint remains in the TB and EB correlation functions. However, in learning the lore regarding the processing of inflationary information, from inflationary graviton to observed CMB photon, I have come up with the approach expressed in the preceding section. The ideas are very rough and need to be worked out in full detail. But if the ideas survive further scrutiny, I believe they could contribute a useful visualization of how inflation phenomenology works. I might even create another pdf on this subject ("Visualizing Inflation Physics").