International Journal of Modern Science and Technology Vol. 1, No. 8, November 2016. Page 264-268.
http://www.ijmst.co/ ISSN: 2456-0235.

Review Article

Survey Report on Space Filling Curves

R. Prethee, A. R. Rishivarman
Department of Mathematics, Theivanai Ammal College for Women (Autonomous)
Villupuram - 605 401. Tamilnadu, India.
*Corresponding author's e-mail: rishivarmanar@gmail.com

Abstract

Space-filling Curves have been extensively used as a mapping from the multi-dimensional space into the one-dimensional space. Space filling curve represent one of the oldest areas of fractal geometry. Mapping the multi-dimensional space into one-dimensional domain plays an important role in every application that involves multidimensional data. We describe the notion of space filling curves and describe some of the popularly used curves. There are numerous kinds of space filling curves. The difference between such curves is in their way of mapping to the one dimensional space. Selecting the appropriate curve for any application requires knowledge of the mapping scheme provided by each space filling curve. Space filling curves are the basis for scheduling has numerous advantages like scalability in terms of the number of scheduling parameters, ease of code development and maintenance. The present paper report on various space filling curves, classifications, and its applications. It elaborates the space filling curves and their applicability in scheduling, especially in transaction.

Keywords: Space filling curve, Holder Continuity, Bi-Measure-Preserving Property, Transaction Scheduling.

Introduction

In mathematical analysis, a space-filling curve is a curve whose range contains the entire 2-dimensional unit square or more generally an n-dimensional unit hypercube. An Space-filling Curves (SFC) acts like a thread that passes through every cell element (or pixel) in the multi-dimensional space so that every cell is visited exactly once. In 1878, George Cantor published a remarkable finding: there exists a bijective function between any two finite-dimensional smooth manifolds. A question that immediately arose in response to Cantor's result was whether or not such a bijection was continuous. In 1879, Netto proved it could not be. After Netto's result, some mathematicians began to look for continuous surjective mappings. In 1890, Peano found one. The Peano curve maps the unit interval into the plane with the image having positive Jordan content. This curve has been called the first space-filling curve (SFC). Other SFCs soon followed with Hilbert's in 1891, Moore's in 1900, Lebesgue's in 1904, Sierpinska's in 1912, and Polya's in 1913 [1]. Despite the creation of these other curves, sometimes space-filling curves are still referred to as Peano curves.

Mathematical tools

The Euclidean Vector Norm

The Euclidean vector norm of semantic vector ϕ_{xi}, also called 2-norm, indicates the length of ϕ_{xi} (Eq. 1) in the Euclidean geometry.

$$||\varphi_{xi}|| = \left(\sum_{k=1}^{n} |\omega^{k}_{xi}|^2\right)^{1/2} = \sqrt{|\omega^1_{xi}|^2 + |\omega^2_{xi}|^2 + \cdots |\omega^n_{xi}|^2} \quad (1)$$

The Semantic Similarity

The semantic similarity (Eq. 2) between two image data objects is defined based on the Euclidean distance between their corresponding semantic vectors. Formally, the semantic similarity between two data objects x_i and x_j is defined as a cosine similarity function.

$$sim(x_i, x_j) = \frac{\varphi_{xi} \cdot \varphi_{xj}}{||\varphi_{xi}||_2 ||\varphi_{xj}||_2} \quad (2)$$

The Quantized Semantic Similarity

Given a set of image data objects $X = \{x_1, x_2, \dots, x_m\}$, find the minimum bounding hyper-rectangle $H(X)$ of the data objects, and divide $H(X)$ into grid cells with a

Received: 17.10.2016; Received after Revision: 20.11.2016; Accepted: 21.11.2016; Published: 23.11.2016

predefined edge length ξ. For any data object x_i, let $\zeta(x_i)$ denote the center of the cell where x_i falls in [2]. The quantized semantic similarity (Eq. 3) can be defined as:

$$sim_Q(x_i, x_j) = \cos^{-1}\frac{\varphi(\zeta(x_i)).\varphi(\zeta(x_j))}{\|\varphi(\zeta(x_i))\|_2 \, \|\varphi(\zeta(x_j))\|_2} \quad (3)$$

Where $\varphi(\zeta(x_i))$ denotes the semantic vector of the center point of the cell where x_i falls in.

The distribution Density

Given the aforementioned data object set X locating in the cells $\pi(H(X))^0, \ldots, \pi(H(X))^{L^n-1}$, the function of distribution density (Eq. 4) within a cell $\pi(H(X))^i$ can be denoted as $F_{dd}(X)^i : R^n \rightarrow R$ which satisfies.

$$\int_{H(X)} F_{dd}(X)^i = 1 \quad (4)$$

Space filling curves

Space-filling curve is a continuous curve in R^2 that passes through every point of the unit square [0, 1]×[0, 1]. A space-filling curve can be thought of as a map from one-dimensional space onto a higher-dimensional space. When working in Euclidean space, it is often thought of as a continuous map from the unit interval [0,1] to R^d $(d > 1)$ whose image has positive Jordan content[3]. Such curves are usually the limit of approximating curves where each approximating curve is bounded, but the length of these curves continues indefinitely. Because the mapping cannot be both continuous and bijective, the final curve is always self-intersecting, even if the approximating curves are not. Two main characteristics of space filling curve are continuous and surjective. It can be shown that if f generates a space-filling curve, then it cannot be bijective.

Classification of space filling curves

Recursive space-filling curve

A space-filling curve $f: I \rightarrow Q \subset R^n$ is called recursive, if both I and Q can be divided in m subintervals and subdomains, such that

- $f*(I^{(\mu)}) = Q^{(\mu)}$

  for all μ= 1;… ;m, and
- All $Q^{(\mu)}$ are geometrically similar to Q.

Contiguous space-filling curve

A recursive space-filling curve is called contiguous, if for any two neighbouring intervals $I^{(v)}$ and $I^{(\mu)}$ also the corresponding subdomains $Q^{(v)}$ and $Q^{(\mu)}$ are direct neighbours, i.e. share an $(n-1)$-dimensional hyper plane [4].

Types of space filling curves

Peano curve

Peano curve (Fig. 1) is the first example of a space-filling curve to be discovered, by Giuseppe Peano in 1890. Peano's curve is a surjective, continuous function from the unit interval onto the unit square, however it is not injective [5]. The first three approximating curves for the Peano curve.

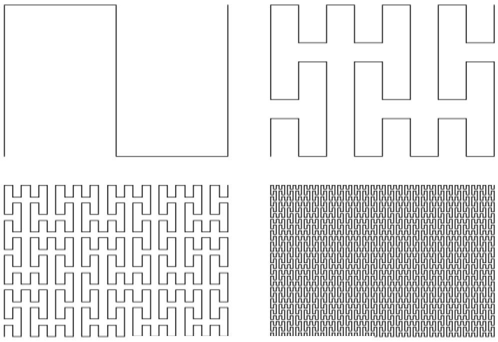

Fig. 1. Peano curve

Hilbert curve

A Hilbert curve (Fig. 2) (also known as a Hilbert space-filling curve) is a continuous fractal space-filling curve first described by the German mathematician David Hilbert in 1891 [6].

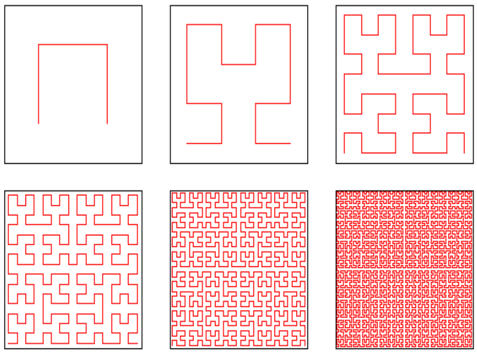

Fig. 2. Hilbert curve

Moore curve

The Moore curve (Fig. 3) is a closed version of the Hilbert curve. It is obtained by concatenating four copies of the Hilbert curve.

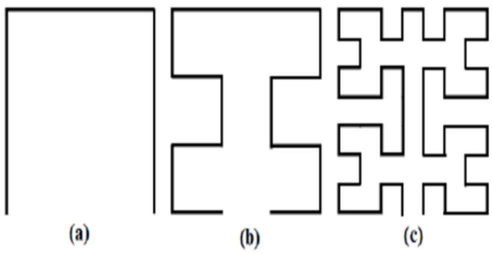

Fig. 3. Moore curve

Lebesgue curve

The Lebesgue curve (Fig. 4) uses a similar subdivision as the Hilbert and Moore curves.

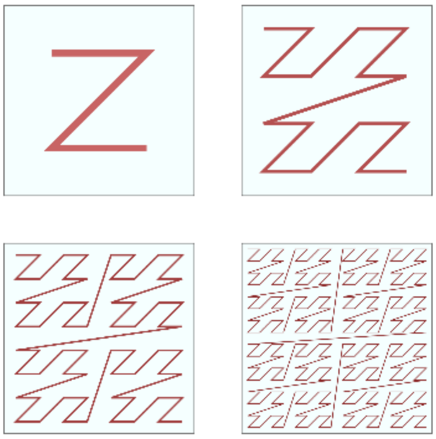

Fig. 4. Lebesgue curve

Sierpinski curve

Sierpinski curves (Fig. 5) are a recursively defined sequence of continuous closed plane fractal curves discovered by Waclaw Sierpinski, which in the limit$(n\rightarrow\infty)$ completely fill the unit square: thus their limit curve, also called the Sierpinski curve, is an example of a space-filling curve. In contrast to the previously discussed space-filling curves the Sierpinski curve is based on a triangular and not a cubical subdivision.

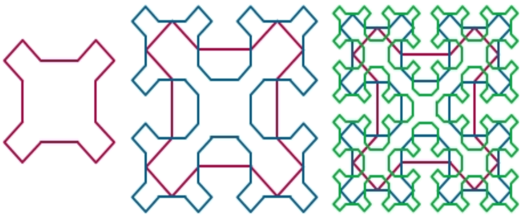

Fig. 5. Seirpinski curve

Gosper curve

The Gosper curve (Fig. 6) is a space filling curve discovered by William Gosper, an American computer scientist, in 1973, and was introduced by Martin Gardner in 1976. The Gosper curve, also known as Peano-Gosper Curve. It is a fractal object similar in its construction to the Hilbert curve.

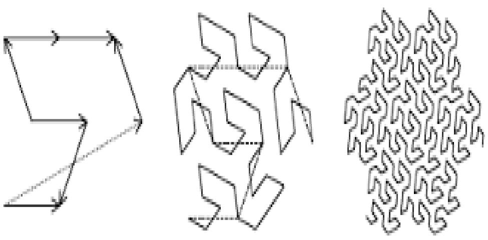

Fig. 6. Gosper curve

Properties of Space-Filling Curve

The important properties of space-filling curves Holder continuity and thebi-measure-preserving property [7].

Holder Continuity

A map$f: I\subset R\rightarrow R^n$ is called Holder continuous of order 1/k (or short Holder-1/k) with Holder constant $c_f$if for all $s;\ t\in I$

$$||f(s)-f(t)|| \leq c_f|s-t|^{\frac{1}{k}} \qquad (5)$$

This property (eq. 5) is also referred to as Lipschitz continuity.

Bi-Measure-Preserving Property

By its recursive construction the Hilbert curve maps an interval to a region with an area equal to the length of the interval. In general we have for d −dimensional Hilbert curves that for any Borelset A ⊂ [0; 1]

$$\lambda_1(A) = \lambda_d(\Psi_d(A)) \qquad (6)$$

Where λ_1 and λ_d denote the one and d-dimensional measure, respectively. This property is called the bi-measure-preserving property.

Mapping in space filling curves

A space-filling curve must be everywhere self-intersecting in the technical sense that the curve is not injective. If a curve is not injective, then one can find two “sub curves” of the curve, each obtained by considering the images of two disjoint segments from the curve’s domain. The two sub curves intersect if the intersection of the two images is non-empty. In general, space-filling curves start with a basic path on a k-dimensional square grid of side 2. The path visits every point in the grid exactly once without crossing itself. It has two free ends, which maybe joined with other paths. The basic curve is said to be of order 1. To derive a curve of order$i$, each vertex of the basic curve is replaced by the curve of order$i$, which may be appropriately rotated and/or reflected to fit the new curve [8].

Applications

Space-filling curves are adopted to define a linear order for sorting and scheduling objects that lie in the multi-dimensional space [9,10]. Using space-filling curves as the basis for scheduling has numerous advantages, like:

- Scalability in terms of the number of scheduling parameters

- Ease of code development and maintenance
- The ability to analyse the quality of the schedules generated and
- The ability to automate the scheduler development process in way similar to automatic generation of programming language compilers [11].

Scheduling transactions using SFC in databases

In a new transaction-scheduling scheme is proposed for real-time database system based on three-dimensional design by integrating the characteristics of value, deadline and criticalness (Fig. 7). Here space-filling curves can be used as they are adopted to define linear order for sorting or scheduling. The space filing curves unnaturally considers value, deadline and criticalness information and gives a scheduling sequence. A CPU request is modelled by multiple parameters, (e.g., the real-time deadline, the criticalness, the priority, etc.) and represented as a point in the multi-dimensional space where each parameter corresponds to one dimension. Using a space-filling curve, the multi-dimensional CPU request is converted to a one dimensional value [12,13].

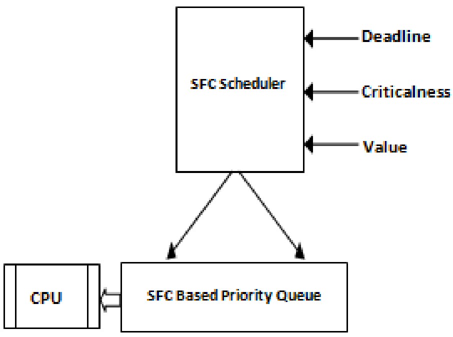

Fig. 7. Space filling curve based CPU scheduler

The space filling curves converts 3-dimensional space using the idea of bit interleaving. The sequence of few transactions obtained after mapping from 3-D to 1-D is shown in table 1.

Table 1. 3-D to 2-D

Point	Dimensions			Bsit	Decimal code
	0	1	2		
(0,1,2)	000	001	010	000001010	10
(2,1,4)	010	001	100	001100010	98
(0,0,7)	000	000	111	001001001	73
(7,0,7)	111	000	111	101101101	365
(7,4,2)	111	100	010	110101100	428

Conclusion

In the present paper, we discussed in detail on the space-filling curves. Space-filling curves are special cases of fractal constructions. Traditionally, such curves have been of interest only to mathematics community and their rendering has been functional. A infinity of space filling curve can be generated, based on square decomposition idea. Space-filling curve techniques have certain unique properties like map the multiple parameters into the one-dimensional space. The different types of space filling curves are analysed. These properties have been used in recently for scheduling CPU transaction. The use of SFC in multi-dimensional feature applications eliminates the distance calculation time and reduces the search time once features are linearly ordered. It can be adopted to define a linear order for sorting and scheduling objects that lie in the multi-dimensional space. As an illustration it is given a unique Space-filling curve technique which can be used in scheduling CPU transaction. Also their mapping and advantages are explored.

Conflict of interest

Authors declare there are no conflicts of interest.

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