Calculus 3 - Vector Functions

In Calculus 1, one application of the derivative is calculating velocity and acceleration. If $s = s(t)$ is position then we found that velocity is

$$v = \frac{ds}{dt} \tag{1}$$

and acceleration is

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \tag{2}$$

For example, with a falling body where $s = 16t^2$ then

$$v = \frac{ds}{dt} = 32t \tag{3}$$

and

$$a = \frac{dv}{dt} = 32 \tag{4}$$

With the introductions of vector functions, we define the velocity vector as

$$\vec{v} = \frac{d\vec{r}(t)}{dt} \tag{5}$$

and acceleration vector as

$$\vec{a} = \frac{d\vec{v}(t)}{dt} \tag{6}$$

Example 1 If $\vec{r} =< t, \frac{1}{2}t^2 >$ then $\vec{r}\,' =< 1, t >$ and so $\vec{v}\,' =< 1, t >$. We also calculate $\vec{a} =< 0, 1 >$

Example 2 If $\vec{r} =< \cos t, \sin t, t >$ then $\vec{r}\,' =< -\sin t, \cos t, 1 >$ so $\vec{v}\,' =< -\sin t, \cos t, 1 >$. We further calculate $\vec{a} =< -\cos t, -\sin t, 0 >$.

Yesterday we defined the unit Tangent and unit Normal vectors and were given by

$$\vec{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}, \quad \vec{N} = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}. \tag{7}$$

For example 1, they were calculated to be

$$\vec{T} = \frac{< 1, t >}{\sqrt{1+t^2}}, \quad \vec{N} = \frac{< -t, 1 >}{\sqrt{1+t^2}}, \tag{8}$$

The are shown in figure 1 (at $t = 1$) as well as the acceleration vector $\vec{a}$.

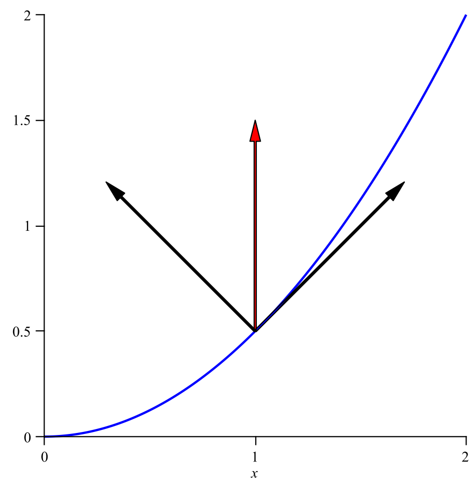

Figure 1: The vectors $\vec{T}$,$\vec{N}$ and $\vec{a}$

It appears that's there a connection between the three. We, in fact, there is and it is as follows:

$$\vec{a} = a_T\vec{T} + a_N\vec{N},$$

where

$$a_T = \frac{d\|\vec{r}\,'\|}{dt}, \quad a_N = \|r'\|\|T'\| \tag{9}$$

Proof

Since

$$\vec{T} = \frac{\vec{r'}}{\|\vec{r}'\|} \tag{10}$$

then

$$\vec{r'} = \|\vec{r}'\|\vec{T} \tag{11}$$

or

$$\vec{v} = \|\vec{r}'\|\vec{T} \tag{12}$$

Differentiating this with respect to t gives

$$\vec{v}' = \|\vec{r}'\|' \ \vec{T} + \|\vec{r}'\| \ \vec{T}' \tag{13}$$

Since

$$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} \tag{14}$$

then (13) becomes

$$\vec{a} = \|\vec{r}'\|'\vec{T} + \|\vec{r}'\|\|\vec{T}'\|\vec{N}. \tag{15}$$

If we define

$$a_T = \|\vec{r}'\|', \qquad a_N = \|\vec{r}'\|\|\vec{T}'\|, \tag{16}$$

then (15) becomes (9). Now since

$$\vec{T} \cdot \vec{T} = 1, \quad \vec{T} \cdot \vec{N} = 0, \quad \vec{N} \cdot \vec{N} = 1 \tag{17}$$

then from (15) we find that

$$a_T = \vec{a} \cdot \vec{T}, \quad a_N = \vec{a} \cdot \vec{N} \tag{18}$$

Example 1

$$\begin{aligned}
\overrightarrow{r} &= \left\langle t, \frac{1}{2}t^2 \right\rangle \\
\overrightarrow{r}' &= \langle 1, t \rangle \\
\|\overrightarrow{r}'\| &= \sqrt{t^2+1}.
\end{aligned}$$

so

$$\overrightarrow{T} = \frac{\overrightarrow{r}'}{\|\overrightarrow{r}'\|} = \left\langle \frac{1}{\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}} \right\rangle$$

Further

$$\begin{aligned}
\overrightarrow{T}' &= \left\langle \frac{-t}{(t^2+1)^{3/2}}, \frac{1}{(t^2+1)^{3/2}} \right\rangle \\
\|\overrightarrow{T}'\| &= \frac{1}{t^2+1}.
\end{aligned}$$

so

$$\overrightarrow{N} = \frac{\overrightarrow{T}'}{\|\overrightarrow{T}'\|} = \left\langle \frac{-t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle$$

The velocity and acceleration are given by

$$\begin{aligned}
\overrightarrow{v} &= \overrightarrow{r}' = \langle 1, t \rangle \\
\overrightarrow{a} &= \overrightarrow{v}' = \langle 0, 1 \rangle.
\end{aligned}$$

$$a_T = \|\overrightarrow{r}'\|' = \frac{t}{\sqrt{t^2+1}}, \qquad a_N = \|\overrightarrow{r}'\|\|\overrightarrow{T}'\| = \frac{1}{\sqrt{t^2+1}}.$$

but we also see that

$$a_T = \vec{a} \cdot \vec{T} = \frac{t}{\sqrt{t^2+1}}, \qquad a_N = \vec{a} \cdot \vec{N} = \frac{1}{\sqrt{t^2+1}}.$$

so

$$\begin{aligned} a_T \vec{T} + a_N \vec{N} &= \frac{t}{\sqrt{t^2+1}} \left\langle \frac{1}{\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}} \right\rangle + \frac{1}{\sqrt{t^2+1}} \left\langle \frac{-t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle \\ &= < 0, 1 > \\ &= \vec{a} \end{aligned}$$

Example 2

$$\vec{r} = \langle \cos t, \sin t, t \rangle \quad \text{so} \quad \vec{r}' = \langle -\sin t, \cos t, 1 \rangle$$

$$\| \vec{r}' \| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}.$$

so

$$\vec{T} = \frac{\vec{r}'}{\| \vec{r}' \|} = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

Further

$$\begin{aligned} \vec{T}' &= \left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \right\rangle \\ \| \vec{T}' \| &= \sqrt{\frac{\sin^2 t}{2} + \frac{\cos^2 t}{2}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

so

$$\vec{N} = \frac{\vec{T}'}{\| \vec{T}' \|} = \langle -\cos t, -\sin t, 0 \rangle$$

The velocity and acceleration are given by

$$\vec{v} = \vec{r}\,' = \langle -\sin t, \cos t, 1\rangle$$
$$\vec{a} = \vec{v}\,' = \langle -\cos t, -\sin t, 0\rangle\,.$$

So

$$a_T = \|\vec{r}\,'\|' = 0, \qquad a_N = \|\vec{r}\,'\|\|\vec{T}\,'\| = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1.$$

Also

$$a_T = \vec{a}\cdot\vec{T} = \langle -\cos t, -\sin t, 0\rangle \cdot \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = 0$$
$$a_N = \vec{a}\cdot\vec{N} = \langle -\cos t, -\sin t, 0\rangle \cdot \langle -\cos t, -\sin t, 0\rangle = 1,$$

so

$$\vec{a} = a_T\vec{T} + a_N\vec{N} = 0\vec{T} + 1\vec{N}.$$