Math 4315 - PDE's Wed. Oct. 12/16

2nd order PDEs

$$a u_{xx} + b u_{xy} + c u_{yy} + lots = 0 \qquad (*)$$

type
$b^2 - 4ac = 0$ parabolic (P)
$b^2 - 4ac > 0$ hyperbolic (H)
$b^2 - 4ac < 0$ elliptic (E)

Target
$u_{ss} + lots = 0$ P
$u_{ss} - u_{rr} + lots = 0$ H ← today
$u_{ss} + u_{rr} + lots = 0$ E

under a change of variable

$r = r(x,y), \quad s = s(x,y)$

For Parabolic

$$a r_x^2 + b r_x r_y + c r_y^2 = 0 \qquad s \text{ - anything}$$

↑

perfect square.

when we sub. our chain rules into (*) ↑
and regroup we obtain

$$(a r_x^2 + b r_x r_y + c r_y^2)\, u_{rr}$$

$$+ \left(2a r_x s_x + b(r_x s_y + r_y s_x) + 2c r_y s_y \right) u_{rs}$$

$$+ (a s_x^2 + b s_x s_y + c s_y^2)\, u_{ss} + \text{lots} = 0$$

To target hyperbolic we need

$$a r_x^2 + b r_x r_y + c r_y^2 = -(a s_x^2 + b s_x s_y + c s_y^2)$$

and $\quad 2a r_x s_x + b(r_x s_y + r_y s_x) + 2c r_y s_y = 0$

too hard right now.

Instead we introduce a modified hyperbolic standard form

$$u_{rs} + \text{lots} = 0$$

Here $\quad a = 0,\ b = 1,\ c = 0 \quad b^2 - 4ac = 1 > 0$ ✓

why - target is easier

$$\left.\begin{aligned} a r_x^2 + b r_x r_y + c r_y^2 &= 0 \\ a s_x^2 + b s_x s_y + c s_y^2 &= 0 \end{aligned}\right\} \text{ these factor.}$$

Ex 1 $u_{xx} - 3u_{xy} + 2u_{yy} = 0$

$b^2 - 4ac = (-3)^2 - 4(1)(2) = 9 - 8 = 1 > 0$ so H

$r_x^2 - 3r_xr_y + 2r_y^2 = 0$ $\quad (r_x - r_y)(r_x - 2r_y) = 0$

$s_x^2 - 3s_xs_y + 2s_y^2 = 0$ $\quad (s_x - s_y)(s_x - 2s_y) = 0$

Pick $r_x - r_y = 0$ $\quad s_x - 2s_y = 0$

M of C $\quad \frac{dx}{1} = \frac{dy}{-1};\ dr = 0$ $\quad \frac{dx}{1} = -\frac{dy}{2};\ ds = 0$

$c_1 = x + y \quad c_2 = r$ $\quad 2x + y = c_1 \quad s = c_2$

$r = R(x+y)$ $\quad s = S'(2x+y)$

Pick $r = x + y,\ s = 2x + y$

$r_x = 1,\ r_y = 1,\ r_{xx} = r_{xy} = r_{yy} = 0$

$s_x = 2 \quad s_y = 1 \quad s_{xx} = s_{xy} = s_{yy} = 0$

chain Rules

$u_{xx} = u_{rr} + 4u_{rs} + 4u_{ss}$

$u_{xy} = u_{rr} + 3u_{rs} + 2u_{ss}$

$u_{yy} = u_{rr} + 2u_{rs} + u_{ss}$

Now sub. into original PDE

$$\cancel{U_{rr}} + 4U_{rs} + 4\cancel{U_{ss}}$$
$$-3(\cancel{U_{rr}} + 3U_{rs} + 2\cancel{U_{ss}})$$
$$+2(\cancel{U_{rr}} + 2U_{rs} + \cancel{U_{ss}}) = 0$$

$\Rightarrow \quad -U_{rs} = 0$ so $U_{rs} = 0$

we can actually integrate this

$U_{rs} = 0 \Rightarrow U_r = f'(r)$ since this is arbitrary let it be the derivative of some function

so $u = f(r) + g(s)$

so back in terms of r & s

$$u = f(x+y) + g(2x+y) \quad \leftarrow \text{general sol}^{n}$$

<u>Ex 2</u> $\quad u_{xx} - 4x^2 u_{yy} + 2u_y = 0$

$b^2 - 4ac = -4(1)(-4x^2) = 16x^2 > 0$ for $x \neq 0$

$$r_x^2 - 4x^2 r_y^2 = 0$$

$$(r_x - 2x r_y)(r_x + 2x r_y) = 0$$

$$r_x - 2x r_y = 0 \quad s_x + 2x s_y = 0$$

$$\frac{dx}{1} = \frac{dy}{-2x}\ ; \quad dr = 0$$

$$2x\,dx = -dy \quad \Rightarrow \quad c_1 = x^2 + y \quad c_2 = r$$

$$\Rightarrow \quad r = R(x^2 + y)$$

Similarily $\quad \frac{dx}{1} = \frac{dy}{2x}\ ; \quad ds = 0$

$$x^2 - y = c_1 \quad c_2 = s$$

$$s = S(x^2 - y)$$

Note

$r = x^2 + y$

$s = x^2 - y$

$r + s = 2x^2$

used later

Pick $\quad r = x^2 + y, \quad s = x^2 - y$

$$r_x = 2x, \quad r_y = 1, \quad r_{xx} = 2 \quad r_{xy} = r_{yy} = 0$$

$$s_x = 2x \quad s_y = -1 \quad s_{xx} = 2 \quad s_{xy} = s_{yy} = 0$$

$$u_y = u_r - u_s$$

$$u_{xx} = 4x^2 u_{rr} + 8x^2 u_{rs} + 4x^2 u_{ss} + 2u_r + 2u_s$$

$$u_{xy} = 2x\,u_{rr} + (0)\,u_{rs} - 2x\,u_{ss}$$

$$u_{yy} = u_{rr} - 2u_{rs} + u_{ss}$$

sub $\quad 4x^2 \cancel{u_{rr}} + 8x^2 u_{rs} + 4x^2 \cancel{u_{ss}} + 2u_r + \cancel{2u_s}$

$$-4x^2 \cancel{u_{rr}} + 8x^2 u_{rs} - 4x^2 \cancel{u_{ss}} + 2u_r - \cancel{2u_s} = 0$$

$$16x^2 u_{rs} + 4u_r = 0 \qquad u_{rs} + \frac{u_r}{4x^2} = 0 \ \text{ or } \ u_{rs} + \frac{u_r}{2(r+s)} = 0$$