Math 2471 - Calc 3

Gradient VF

$$\bar{F} = \bar{\nabla} f$$

$$= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$$

or $\bar{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

f - potential function

If we are given

$$\bar{F} = \langle f, g \rangle$$

how do we know if f exists such that $\bar{F} = \bar{\nabla} f$?

If so, we say the VF is "conservative."

so is $\bar{F} = \langle 3x^2y - 3y^2 + 1,\ x^3 - 6xy \rangle$ conservative

if so $\bar{F} = \bar{\nabla} f$ for some f

so $f_x = 3x^2y - 3y^2 + 1$ so $f_{xy} = 3x^2 - 6y$

$f_y = x^3 - 6xy$ & $f_{yx} = 3x^2 - 6y$

and they are the same so f exists!

$f_x = 3x^2y - 3y^2 + 1 \Rightarrow f = x^3y - 3xy^2 + x + A(y)$

$f_y = x^3 - 6xy \Rightarrow f = x^3y - 3xy^2 + B(x)$

so $f = x^3y - 3xy^2 + x + c$

what about going to 3D.

Monday

$\bar{F} = \langle yz^2 + y,\ xz^2 + x,\ 2xyz - 3 \rangle$ (4)

we introduced the "Del operator"

$$\bar{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

& Divergence of VF $\quad F = \langle f, g, h \rangle$

$$\bar{\nabla} \cdot \bar{F} = \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g + \frac{\partial h}{\partial z}$$

& curl of VF

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

we were able to show that

$\bar{\nabla} \times \bar{F} = \bar{0}$ for the VF (*)

if $\bar{\nabla} \times \bar{F} = \bar{0}$ then the VF is conservative

& f exists!

so $f_x = yz^2 + y \Rightarrow f = xyz^2 + xy + A(y,z)$

$f_y = xz^2 + x \Rightarrow f = xyz^2 + xy + B(x,z)$

$f_z = 2xyz - 3 \Rightarrow f = xyz^2 - 3z + C(x,y)$

choose $A = B = -3z, \quad C = xy$ so

$f = xyz^2 + xy - 3z + c \qquad c$ arb const.

Ex is $\bar{F} = \langle 3x^2, z^2, x^3 + 2yz \rangle$ cons.?

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & z^2 & x^3 + 2yz \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^3 + 2yz \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 3x^2 & x^3 + 2yz \end{vmatrix} j + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 3x^2 & z^2 \end{vmatrix} \bar{k}$$

$$= (2z - 2z)\vec{i} - (3x^2 - 0) j + (0, 0)\bar{k}$$

$= \langle 0, -3x^2, 0\rangle \neq \langle 0,0,0\rangle$

so not conservative. Let's try anyway

$f_x = 3x^2 \Rightarrow f = x^3 + A(y,z)$

$f_y = z^2 \Rightarrow f = yz^2 + B(x,z)$

$f_z = x^3 + 2yz \Rightarrow f = x^3 z + yz^2 + C(x,y)$

Consider piece

$f = x^3$ choose $B(x,z)$ & $C(x,y)$ to get this

yz^2 choose A to get this

$x^3 z$ $A(y,z)$ cannot get this piece

Ex $\quad \bar{F} = \langle z\sec^2 x,\ z,\ y+\tan x \rangle$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z\sec^2 x & z & y+\tan x \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & y+\tan x \end{vmatrix} i - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z\sec^2 x & y+\tan x \end{vmatrix} j$$

$$+ \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z\sec^2 x & z \end{vmatrix} k$$

$$= (1-1)\,i - (\sec^2 x - \sec^2 x)\,j + 0\,k = \bar{0} \checkmark$$

$f_x = z\sec^2 x \Rightarrow f = z\tan x + A(y,z)$

$f_y = z \Rightarrow f = yz + B(x,z)$

$f_z = y+\tan x \Rightarrow f = yz + \tan x\, z + C(x,y)$

$$f = z\tan x + yz + c$$