Math 6345 AODEs

Last class we considered

$$\dot{\bar{x}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bar{x}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad A^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \quad A^3 = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \quad A^4 = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} t + \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \frac{t^3}{3!} + \cdots$$

$$= \begin{pmatrix} 1 + 2t + \frac{5t^2}{2!} + \frac{14t^3}{3!} + \cdots & t + \frac{4t^2}{2!} + \frac{13t^3}{3!} + \cdots \\ t + \frac{4t^2}{2!} + \frac{13t^3}{3!} + \cdots & 1 + 2t + \frac{5t^2}{2!} + \frac{14t^3}{3!} + \cdots \end{pmatrix}$$

However, we really don't know what this looks like

Now, we saw

$$e^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} t} = e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t}$$

$$= e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t} \cdot e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t}$$

are these easier to calculate?

First

$$\dot{\bar{x}} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \bar{x}$$

$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad A^2 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}$$

so $\quad A^n = \begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix}$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} t + \begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix} \frac{t^2}{2!} + \cdots$$

$$= \begin{pmatrix} 1 + at + \frac{a^2t^2}{2!} + \cdots & 0 \\ 0 & 1 + dt + \frac{d^2t^2}{2!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{at} & 0 \\ 0 & e^{dt} \end{pmatrix}$$

so $\quad e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}$

if $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So $e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{t^3}{3!} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{t^4}{4!} + \cdots$

$$= \begin{matrix} 1 + t^2/2! + \frac{t^4}{4!} + \cdots & t + t^3/3! + t^5/5! \\ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \end{matrix}$$

Now $\quad e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$

$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}$

$$e^{At} = \begin{pmatrix} \frac{e^{t} + e^{-t}}{2} & \frac{e^{t} - e^{-t}}{2} \\ \frac{e^{t} - e^{-t}}{2} & \frac{e^{t} + e^{-t}}{2} \end{pmatrix}$$

So $$e^{\begin{pmatrix}2&1\\1&2\end{pmatrix}t} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t}\end{pmatrix}\begin{pmatrix} \frac{e^{t}+e^{-t}}{2} & \frac{e^{t}-e^{-t}}{2} \\ \frac{e^{t}-e^{-t}}{2} & \frac{e^{t}+e^{-t}}{2}\end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{3t}+e^{t}}{2} & \frac{e^{3t}-e^{t}}{2} \\ \frac{e^{3t}-e^{t}}{2} & \frac{e^{3t}+e^{t}}{2}\end{pmatrix}$$

and you can check that a Taylor expansion of this gives what we have on the 1st page.

So now we question whether

$$e^{(A+B)t} = e^{At} \cdot e^{Bt}$$

with the following example

$$e^{\begin{pmatrix}1&1\\0&0\end{pmatrix}} = e^{\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}} = e^{\begin{pmatrix}1&0\\0&0\end{pmatrix}} \cdot e^{\begin{pmatrix}0&1\\0&0\end{pmatrix}}$$

1st $e^{\begin{pmatrix}1&1\\0&0\end{pmatrix}t}$

Now $\begin{pmatrix}1&\cancel{1}\\0&0\end{pmatrix}^2 = \begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}1&1\\0&0\end{pmatrix}$

so $\begin{pmatrix}1&1\\0&0\end{pmatrix}^n = \begin{pmatrix}1&1\\0&0\end{pmatrix}$

$$e^{\begin{pmatrix}1&1\\0&0\end{pmatrix}t} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&1\\0&0\end{pmatrix}t + \begin{pmatrix}1&1\\0&0\end{pmatrix}t^2/2! + \cdots$$

$$= \begin{pmatrix}1+t+t^2/2!+\cdots & t+t^2/2!+t^3/3! \\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}e^t & e^t-1 \\ 0 & 1\end{pmatrix}$$

Next $e^{\begin{pmatrix}1&0\\0&0\end{pmatrix}t}$

Now $\begin{pmatrix}1&0\\0&0\end{pmatrix}^2 = \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

and so $\begin{pmatrix}1&0\\0&0\end{pmatrix}^n = \begin{pmatrix}1&0\\0&0\end{pmatrix}$

$$\Rightarrow \quad e^{\begin{pmatrix}1&0\\0&0\end{pmatrix}t} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&0\end{pmatrix}t + \begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{t^2}{2!} + \cdots$$

$$= \begin{pmatrix}1+t+\frac{t^2}{2!}+\cdots & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}e^t & 0\\ 0 & 1\end{pmatrix}$$

Next $\quad e^{\begin{pmatrix}0&1\\0&0\end{pmatrix}t}$

$\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ so the rest of $= 0$

$$e^{\begin{pmatrix}0&1\\0&0\end{pmatrix}t} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}t + [0]$$

$$= \begin{pmatrix}1&t\\0&1\end{pmatrix}$$

so $e^{\begin{pmatrix}1&0\\0&0\end{pmatrix}t} \cdot e^{\begin{pmatrix}0&1\\0&0\end{pmatrix}t}$

$$= \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & te^t \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} e^t & e^t - 1 \\ 0 & 1 \end{pmatrix}$$

so here is an example when

$$e^{(A+B)t} \neq e^{At} \cdot e^{Bt}$$

so what is special about

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

& not $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

so what is special is that

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ & $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ commute under multiplication

& $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ do not

check $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ > not the same.

Here we now prove that

$$e^{(A+B)t} = e^{At} \cdot e^{Bt}$$

if $AB = BA$

L.S. $e^{(A+B)t} = I + (A+B)t + (A+B)^2 \frac{t^2}{2!} + (A+B)^3 \frac{t^3}{3!} + \dots$

R.S. $e^{At} \cdot e^{Bt} = \left(I + At + \frac{A^2t^2}{2!} + \dots\right)\left(I + Bt + \frac{B^2t^2}{2!} + \dots\right)$

mult RHS out

$$= I + Bt + \frac{B^2t^2}{2!} + \frac{B^3t^3}{3!} + \cdots$$

$$\cancel{A}\, At + ABt^2 + AB^2\frac{t^3}{2!}$$

$$+ \frac{A^2}{2!}t^2 + \frac{A^2B}{2!}t^3 + \cdots$$

$$+ \frac{A^3}{3!}t^3 + \cdots$$

$$= I + (A+B)t + \left(\frac{A^2}{2!} + AB + \frac{B^2}{2!}\right)t^2$$

$$+ \frac{A^3}{3!} + \frac{A^2B}{2!} + \frac{AB^2}{2!} + \frac{B^3}{3!}$$

Now $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$

$= A^2 + 2AB + B^2$

= thus the coeff of t^2 chang

similarly

$$(A+B)^3 = (A+B)(A^2+AB+BA+B^2)$$

$$= A^3 + A^2B + ABA + AB^2$$
$$+ BA^2 + BAB + B^2A + B^3$$

$$= A^3 + 3A^2B + 3AB^2 + B^2$$

which is coeff of t^3 (divide by 3!)

thus we can prove for n

How, still knowing that we can split up the matrix

Ex $\dot{\vec{x}} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x}$ How to split $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

maybe $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$ or $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$?

or $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ but do they commute?