# Math 6378 – Sym

The last 2 classes we consider invariance and trans. that lead to separable ODE.

Namely

Ex1 $$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{x^3 y} + \frac{1}{x^2} \Rightarrow \frac{ds}{dr} = \frac{-r^2 + r + 1}{2r^3 + r^2 + r}$$

L.G. $\bar{x} = e^{\varepsilon} x$, $\bar{y} = e^{-\varepsilon} y$  TX $x = re^{s}$, $y = e^{-s}$

Ex2 $$\frac{dy}{dx} = \frac{y}{x} + \frac{y^3}{x^4} \Rightarrow \frac{ds}{dr} = -\frac{1}{r^3}$$

LG. $\bar{x} = \frac{x}{1+\varepsilon x}$, $\bar{y} = \frac{y}{1+\varepsilon x}$  TX $x = \frac{1}{s}$, $y = \frac{r}{s}$

Ex3 $$\frac{dy}{dx} = \frac{y^2}{xy + x^3} \Rightarrow \frac{ds}{dr} = \frac{1}{r^3}$$

LG

$\bar{x} = \frac{(y+\varepsilon)x}{y}$, $\bar{y} = y + \varepsilon$  TX $x = rs$, $y = s$

Note: Each sep. ODE inv. $\bar{r} = r$, $\bar{s} = s + \varepsilon$.

Now each transformation is invariant under the LG $\epsilon_1$ $LG_2 = \{ \bar{r} = r, \bar{s} = s + \varepsilon \}$

To illustrate

Consider

$$TX = \quad x = \frac{1}{s}, \quad y = \frac{r}{s}$$

$$LG_1 = \left\{ \frac{x}{1+\varepsilon x}, \frac{y}{1+\varepsilon x} \right\} \qquad LG_2 = \{ \bar{r} = r, \bar{s} = s + \varepsilon$$

so $\bar{x} = \dfrac{1}{\bar{s}} \Rightarrow \dfrac{x}{1+\varepsilon x} = \dfrac{1}{s+\varepsilon} \Rightarrow x(s+\varepsilon) = 1 + \varepsilon x$

$$x s + \cancel{\varepsilon x} = 1 + \cancel{\varepsilon x}$$

$\Rightarrow \quad x = \dfrac{1}{s}$ ✓

and $\bar{y} = \dfrac{\bar{r}}{\bar{s}} \Rightarrow \dfrac{y}{1+\varepsilon x} = \dfrac{r}{s+\varepsilon}$

$$\Rightarrow \quad y(s+\varepsilon) = r(1+\varepsilon x)$$

$$\cancel{ys + \varepsilon y = r + \varepsilon x r}$$

$$y(s+\varepsilon) = r\left(1 + \frac{\varepsilon}{s}\right) = \frac{r\,(s+\varepsilon)}{s} \Rightarrow y = \frac{r}{s} \checkmark$$

So given both LG's could we find the transformation.

Consider

$$x = A(r, s) \quad \& \quad y = B(r, s)$$

and let's see if we can find $A$ & $B$

ex 1

$$\bar{x} = A(\bar{r}, \bar{s}) \Rightarrow x = A(r, s)$$

so $$e^{\varepsilon} x = A(r, s+\varepsilon)$$

$$\Rightarrow \boxed{e^{\varepsilon} A(r, s) = A(r, s+\varepsilon)}$$

$$\bar{y} = B(\bar{r}, \bar{s})$$

so $$e^{-\varepsilon} y = B(r, s+\varepsilon)$$

or $$\boxed{e^{-\varepsilon} B(r, s) = B(r, s+\varepsilon)}$$

Two functional equations

so $\quad e^{-\varepsilon} A(r, s+\varepsilon) = A(r,s)$

$$e^{\varepsilon} B(r, s+\varepsilon) = B(r,s)$$

must be absent of $\varepsilon$

so $\dfrac{\partial}{\partial \varepsilon} e^{-\varepsilon} A = 0 \Rightarrow -e^{-\varepsilon} A + e^{-\varepsilon} A_2 = 0$

$$\dfrac{\partial}{\partial \varepsilon} e^{\varepsilon} B = 0 \Rightarrow e^{\varepsilon} B + e^{\varepsilon} B_2 = 0$$

cancel $\quad e^{-\varepsilon} \left(e^{\varepsilon}\right)$ & set $\varepsilon = 0$

so $\quad A_s - A = 0 \quad B_s + B = 0$

$$A = a(r)\, e^{s}, \quad B = b(r)\, e^{-s}$$

if we choose

$$a(r) = r \quad b(r) = 1$$

gives

$$A = r e^{s}, \quad B = e^{-s}$$

the TX we had.

Ex2 $\quad x = A(r,s) \qquad y = B(r,s)$

so $\quad \bar{x} = A(\bar{r},\bar{s}) \Rightarrow \dfrac{x}{1+\varepsilon x} = A(r, s+\varepsilon)$

$$\Rightarrow \quad x = (1+\varepsilon x)\, A(r, s+\varepsilon)$$

& $\quad \bar{y} = B(\bar{r},\bar{s}) \Rightarrow \dfrac{y}{1+\varepsilon x} = B(r, s+\varepsilon)$

$$y = (1+\varepsilon x)\, B(r, s+\varepsilon)$$

$$\left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$$

so $\quad x A(r, s+\varepsilon) + (1+\varepsilon x) A_2(r, s+\varepsilon) = 0$

$\varepsilon = 0 \qquad A A + A_s = 0$

so $\quad -\dfrac{dA}{A^2} = ds \Rightarrow \dfrac{1}{A} = s + a(r)$

$$A = \frac{1}{s + a(r)}$$

Similarly

$$(1+\varepsilon x)\, B_2 + xB = 0$$

$\varepsilon = 0$ $$B_{\Sigma} + AB = 0$$

$$\frac{dB}{B} = -A = -\frac{1}{s+a(r)}$$

$$\ln B = -\ln s+a(r) + \ln b(r)$$

so $$B = \frac{b(r)}{s+a(r)}$$

so $$A = \frac{1}{s+a(r)} \qquad B = \frac{b(r)}{s+a(r)}$$

Set $a(r) = 0$, $b(r) = r$

$$\Rightarrow \quad x = A = \frac{1}{s}, \quad y = B = \frac{r}{s}$$

as we saw earlier.

However, we need to have the LG in hand!

$$\bar{x} = f(x, y, \varepsilon), \quad \bar{y} = g(x, y, \varepsilon)$$

$$\downarrow$$

$$\frac{d\bar{y}}{d\bar{x}} = F(\bar{x}, \bar{y}) \underset{LG}{\Rightarrow} \frac{dy}{dx} = F(x, y)$$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d}{dx} g}{\frac{d}{dx} f} = \frac{g_x + g_y y'}{f_x + f_y y'} = F\left( f(x,y), g(x,y) \right)$$

$$\Rightarrow \quad g_x + g_y y' = F f_x + F f_y y'$$

$$\Rightarrow \quad y' = \frac{F(f,g) f_x - g_x}{g_y - F(f,g) f_y} = F$$

a very complicated eqn for $f$ & $g$!

# Infinitesimal LG.

given $\bar{x} = f(x, y, \varepsilon)$, $\bar{y} = g(x, y, \varepsilon)$

perform a Taylor expansion in $\varepsilon$ to $O(\varepsilon^2)$

<u>Note:</u> $\varepsilon = 0$ $\bar{x} = x$, $\bar{y} = y$

so $$\bar{x} = x + \left.\frac{\partial f}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon + O(\varepsilon^2)$$

$$\bar{y} = y + \left.\frac{\partial g}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon + O(\varepsilon^2)$$

& denote

$$X(x, y) = \left.\frac{\partial f}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad Y(x, y) = \left.\frac{\partial g}{\partial \varepsilon}\right|_{\varepsilon=0}$$

$X, Y$ are called "infinitesimals".

Ex1 L.E. 1 $\quad \bar{x} = e^{\varepsilon} x, \quad \bar{y} = e^{-\varepsilon} y$

$e^{\varepsilon} = 1 + \varepsilon + \varepsilon^2 \ldots \qquad e^{-\varepsilon} = 1 - \varepsilon + \varepsilon^2 - \ldots$

$$\bar{x} = (1 + \varepsilon + O(\varepsilon^2))x$$

$$= x + x\varepsilon + O(\varepsilon^2)$$

$$\bar{y} = (1 - \varepsilon + O(\varepsilon^2))y$$

$$= y - y\varepsilon + O(\varepsilon^2)$$

So $X = x, \quad Y = -y$

Ex2 $\quad \bar{x} = \dfrac{x}{1+\varepsilon x}, \quad \bar{y} = \dfrac{y}{1+\varepsilon x}$

$$\frac{\partial f}{\partial \varepsilon} = \frac{-x^2}{(1+\varepsilon x)^2} \qquad X = -x^2$$

$$\frac{\partial y}{\partial \varepsilon} = \frac{-xy}{(1+\varepsilon x)^2} \qquad Y = -xy$$

Q:

(1) Given the infinitesimals how hard (easy) is it to find r's s?

(2) How hard (easy) is it to find X & Y?