Robust Graph Topologies for Networked Systems

Waseem Abbas * **Magnus Egerstedt** *

* *Georgia Institute of Technology, Atlanta, GA 30332 USA*
(e-mails: wabbas@gatech.edu, magnus@gatech.edu).

Abstract: Robustness of networked systems against noise corruption and structural changes in an underlying network topology is a critical issue for a reliable performance. In this paper, we investigate this issue of robustness in networked systems both from structural and functional viewpoints. Structural robustness deals with the effect of changes in a graph structure due to link or edge failures, while functional robustness addresses how well a system behaves in the presence of noise. We discuss that both of these aspects are inter-related, and can be measured through a common graph invariant. A graph process is introduced where edges are added to an existing graph in a step-wise manner to maximize robustness. Moreover, a relationship between the symmetry of an underlying network structure and robustness is also discussed.

Keywords: Networked systems, Graph theoretic models, Robustness, Linear consensus.

1. INTRODUCTION

Robustness in networked systems can be studied from two different perspectives. Firstly, how well a system behaves in the presence of noise, i.e. robustness against noise or functional robustness, and secondly what is the effect of change in network topology (due to edge or node failures) on the performance of such systems, i.e., structural robustness. Both of these aspects have been studied in the literature and various indices have been proposed to measure them. Edge (vertex) connectivity, algebraic connectivity as introduced in Fiedler (1973), betweenness discussed in Freeman (1977), information centrality, toughness and other spectral measures (see Wu et al. (2011)) are some of the parameters that have been used to quantify structural robustness in graph structures. Robustness of networks where agents implement consensus protocols in the presence of noise has been addressed by providing various distributed algorithms and schemes to minimize corruption of noise in such systems. Examples include Xiao et al. (2007), Wang et al. (2009) and Young et al. (2010). Most of the studies on structural robustness and robustness against noise seem to be independent of each other, focusing either one of the aspects. Here, we show that both of these robustness viewpoints are in fact, related to each other and therefore, can be measured simultaneously by a same parameter.

A network of agents can be modelled by an undirected graph where vertices represent agents and edges are the information exchange links among agents. Recently, Young et al. (2010) and Young et al. (2011) has shown that functional robustness of systems, where agents update their states by a linear consensus protocol in the presence of additive white noise, can be measured by a so called *Kirchhoff index* of a graph. On the other hand, Ellens et al. (2011) has shown that the effect of edge failures on the overall connectivity of a graph can be quantified by an effective graph resistance, which is equivalent to the Kirchhoff index of a graph (as shown in Klein and Randić (93)). Thus, both aspects of robustness can be specified by an exactly same graph invariant.

Klein and Randić (93) introduced the Kirchhoff index of a graph through the notion of effective graph resistance. An electrical network can be obtained from a graph by replacing each edge with a unit resistance. The total electrical resistance between any two nodes in such a network is the effective resistance between the corresponding vertices of a graph. The sum of effective resistance between any two vertices is the Kirchhoff index, K_f, or the effective resistance of a graph (see Ellens et al. (2011)).

In this paper, we further explore this relationship between structural robustness and functional robustness (robustness due to noise) in multiagent systems. The paper proposes to unify these two notions of robustness through the concept of Kirchhoff index of the underlying network topology. We also investigate the role of various network topologies on the robustness property of these systems. In particular, Kirchhoff indices of some special families of graphs are computed, and these calculations are used to obtain a greedy algorithm for adding edges in a graph to maximize its robustness. Moreover, a relationship between the symmetry of a network structure and its robustness is also discussed.

2. ROBUSTNESS ISSUES IN NETWORKED SYSTEMS

Agents exchange information with each other locally in distributed systems. This exchange of information is possible through an interconnection network of agents that can be modelled by a graph structure. For example, agents agree on a common value (that may be a sensor measurement) by implementing a linear consensus protocol. In fact, connectivity of the underlying graph structure is a necessary requirement for the consensus protocol to work (see Mesbahi and Egerstedt (2010) as an example). Moreover, the structure of the underlying network affects various properties of a system including convergence rates, connectivity of the network under edge (interconnection among agents) or vertex (agent) failures. A highly connected network is obviously less affected by an edge or vertex failures and is therefore, more *robust* to these deletions. Thus, the structure of the interconnection infrastructure plays a key role when understanding the effects of edge or vertex failures.

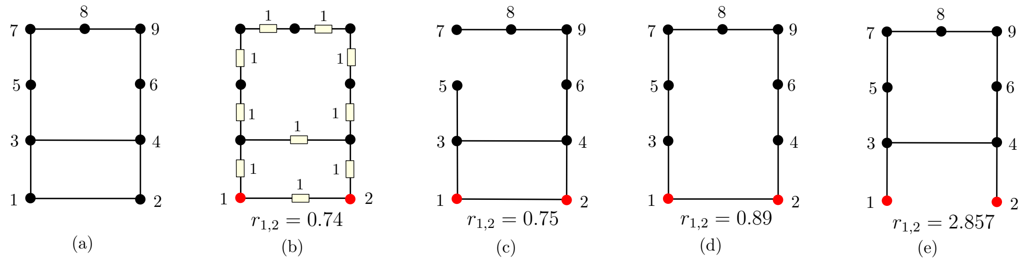


Fig. 1. (a) A graph with nine nodes. (b) Each edge is replaced by a unit resistance and the effective resistance between the nodes 1 and 2 is calculated. In each of (c), (d) and (e), an edge is lost resulting in a loss of path between nodes 1 and 2. A corresponding increase in $r_{1,2}$ is also shown. Note that a smaller $r_{1,2}$ indicates a more robust connection between the nodes 1 and 2.

Another aspect of robustness comes into play when we also consider the agents' dynamics in such systems. These agents compute their states (that may be their positions or any other measurements) and eventually exchange them with others through some medium that may be noisy. This noise plays an important role in determining the overall functionality of the system. It has been observed that some network topologies are least affected by the incorporation of noise when agents are performing linear consensus, while others are affected to a larger extent (see Young et al. (2011) for example). The network structures minimally affected by noise are obviously more *robust*. This leads us towards two aspects of *robustness* in multiagent systems where agents run the consensus dynamics.

(a) *Structural Robustness:* It is the ability of the network to maintain its original structure and inter-connection among vertices in the underlying graph under edge or vertex failures.

(b) *Funtional Robustness:* It measures how well a system behaves in the presence of noise that corrupts measurements or an information exchange among agents.

The above mentioned robustness views seem to have a different focus, where (a) is related purely to a property of the underlying graph structure while (b) deals with the effect of noise on measurements and states of the agents. We show here that both these robustness views are in fact, related to each other and can be measured by the same parameter.

2.1 Structural Robustness vs. Functional Robustness

There may exist multiple paths between two nodes in a given graph of a network. A large number of unique paths between two nodes implies that these nodes are highly interconnected with each other. Thus, their connectivity with each other will not be effected to a large extent by an edge failure, indicating a robust connection between these nodes. The number of unique paths between any two nodes, therefore, hints upon the quantitative aspect of structural robustness in a network.

It is not only the number of unique paths, but also the $quality$ of paths that is crucial to the robustness against edge failures. A path of shorter length between two nodes is preferred over a longer one as it corresponds to an increased level of connectedness between these nodes due to a lesser delay. Also, shorter length paths between nodes result in short random walks that are less affected by the node or edge failures as shown in Chandra et al. (1996).

Thus, the structural robustness should incorporate both the quantitative as well as the qualitative effect of edge removals on the overall connectivity of the network. As it is shown in Ellens et al. (2011), the notion of *effective resistance* between two nodes takes into account both of these aspects, i.e., the number of paths between two nodes and the length of these paths. Effective resistance between nodes decreases with an increase in the number of paths between nodes. Also, the effective resistance between nodes is smaller if the length of the paths between them is shorter. This provides a nice way to quantify the structural robustness in networks.

The *effective resistance* between any two vertices i and j in an un-weighted graph G is denoted by $r_{i,j}$. It is defined as the effective electrical resistance between the points i and j when a resistor of unit resistance is placed along every edge and a potential difference is applied between i and j as illustrated in the Fig. 1. Consider a network in the Fig. 1. There are three unique paths between the nodes 1 and 2, namely $x = [1 \to 2]$, $y = [1 \to 3 \to 4 \to 2]$ and $z = [1 \to 3 \to 5 \to 7 \to 8 \to 9 \to 6 \to 4 \to 2]$. Each of these paths adds to the robustness of connection between nodes 1 and 2. Since, path z is the longest one, it has a least contribution towards the robustness of inter-connection between nodes 1 and 2. This is also indicated by only a slight increase in the $r_{1,2}$ value in Fig. 1(c), where the loss of an edge $5 \sim 7$ results in the loss of z path between 1 and 2. Similarly, when a path y is lost, $r_{1,2}$ is increased to a greater degree as y path has a shorter length than z. When the shortest (most crucial) path, x, is lost, the greatest increase in $r_{1,2}$ is observed.

Thus, structural robustness of the overall network having n nodes can be measured by the sum of the effective resistances over all pairs of nodes in the underlying graph, which is the so called *Kirchhoff index*, K_f, of the graph.

$$K_f(G) = \sum_{1 \leq i < j \leq n} r_{i,j} \tag{1}$$

Here, $r_{i,j}$ is an effective resistance between nodes i and j.

A smaller value of K_f, indicates that a network is structurally more robust. It is also interesting to see that the addition of an edge strictly decreases the value of K_f in a graph (shown in Ellens et al. (2011)), thus, increasing robustness. This also supports our intuition as addition of an edge will always result in an extra path between a pair of nodes.

For the case of network robustness against noisy measurements, i.e. *functional robustness*, we consider a multiagent system with agents implementing a linear consensus protocol. Linear consensus dynamics has been extensively studied in the domain of network control systems due to its wide variety of applications including formation control, distributed control mechanisms, sensor networks and cooperative decision making to name a few (see Mesbahi and Egerstedt (2010)). Simple consensus dynamics of such a system can be given as,

$$\dot{x}(t) = -Lx(t) \tag{2}$$

where L is a laplacian matrix of an underlying graph and x is a corresponding state vector of the agents. In steady state, agents reach an agreement over a common state $\bar{x}(t)$. But for practical systems, agents' states are affected by a noise term. Thus,

$$\dot{x}(t) = -Lx(t) + \xi(t) \tag{3}$$

where $\xi(t)$ is a zero-mean mutually white stochastic process. It is known (e.g. see Xiao et al. (2007) and Young et al. (2010)) that in the presence of this noise term, agents states do not converge to a common value but will remain in motion about $\bar{x}(t)$. In Young et al. (2010), robustness of a system in (3) under noisy consensus dynamics is then defined in terms of the expected dispersion of the system from consensus. A nice result reported there relates this robustness due to noisy consensus under the above setting to the *Kirchhoff index* of the undirected graph structure of the underlying network. It is shown that a network with a greater Kirchhoff index has a greater dispersion from consensus due to noise and is therefore, less robust. Similarly, a smaller value of K_f indicates that the expected dispersion of the system in (3) due to noise is not significant, thus, indicating a greater robustness of network against noise.

In the light of the above discussion, it can be stated that seemingly different notions of structural robustness and functional robustness are in fact, very inter-related. Both of them depend on the structure of an underlying network and can be measured by a same graph invariant known as the Kirchhoff index.

3. KIRCHHOFF INDEX OF SOME GRAPHS

As already discussed above that Kirchhoff index can measure both structural and functional robustness in multiagent systems. This provides us a way to develop a systematic scheme for designing optimal network topologies to maximize their robustness properties. In Section 4, we introduce a graph process where a single edge is added to an existing graph at each step to minimize the Kirchhoff index. In this section, we present some results that will be used to obtain a greedy algorithm for adding edges in a graph to maximize robustness. We find the Kirchhoff index of various graph structures and also present optimal[1] addition of edges for some specific graphs. At first, some graph terminologies are introduced.

A *Star Graph*, $\mathcal{S}_m$, is a tree with m vertices where $m-1$ vertices have a degree 1 and they all are connected to a single central vertex that has a degree $m-1$. A *Fan Graph*, $\mathcal{F}_m$ is obtained by connecting all the vertices in a path graph, $\mathcal{P}_{m+1}$, to a single vertex as shown in the Fig. 2.

Let G_1 and G_2 be two graphs. Let u and v be the maximum degree vertices of G_1 and G_2 respectively. We use a notation $G_1 \bullet G_2$ to denote a graph obtained by identifying $u \in G_1$ with a vertex $v \in G_2$. An example is shown in the Fig. 2. Also, $(G_1)^k$ will be referred to denote a graph obtained from k copies of G_1 by identifying their maximum degree vertices, e.g., $(G_1)^3 = G_1 \bullet (G_1 \bullet G_1)$.

We also refer to $\mathcal{F}_i$ as an *i-petal*, and $(\mathcal{F}_i)^k$ as a *petal graph* containing k number of i-petals. An example is illustrated in the Fig. 2.

Lemma 1. The Kirchhoff index of $G = (\mathcal{F}_1)^k$ is

$$K_f((\mathcal{F}_1)^k) = \frac{2}{3}k(4k-1) \tag{4}$$

[1] in the sense of minimizing the Kirchhoff index, K_f.

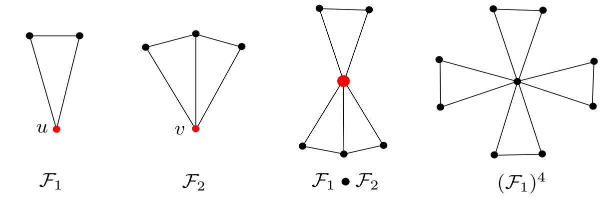


Fig. 2. Fan graphs $\mathcal{F}_1$ and $\mathcal{F}_2$. Note that $\mathcal{F}_2$ is obtained by connecting all the vertices of a path graph with three nodes, $\mathcal{P}_3$, to a common node v. $\mathcal{F}_1 \bullet \mathcal{F}_2$ is obtained by identifying u and v vertices in $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. A petal graph, $(\mathcal{F}_1)^4$, with four 1-petals is also shown.

Proof. There are $2k+1$ vertices in $(\mathcal{F}_1)^k$. We label its vertices as $\{1, 2, \cdots, 2k, \alpha\}$, where α is the central vertex with a maximum degree as shown in the Fig. 3. Note that if i is odd, $r_{i,i+1} = 2/3$ and $r_{i,j} = 4/3$ for every $j > i+1$. For even i, $r_{i,j} = 4/3$ for every $j > i$. Thus, for a fixed i,

$$\sum_{i<j} r_{i,j} = \begin{cases} \frac{4}{3}(2k-i) & i \text{ is even} \\ \frac{2}{3} + \frac{4}{3}(2k-i-1) & i \text{ is odd} \end{cases}$$

Also, for every $i \in \{1, 2, \cdots, 2k\}$, we have $r_{i,\alpha} = 2/3$. Thus, Kirchhoff index of $(\mathcal{F}_1)^k$ can be written as,

$$K_f((\mathcal{F}_1)^k) = \sum_i r_{i,\alpha} + \sum_{i,j>i} r_{i,j}$$

After inserting the values and simplification we get,

$$K_f((\mathcal{F}_1)^k) = \frac{2}{3}(2k) + \left[\frac{8}{3}k(k-1) + \frac{2}{3}k\right] = \frac{2}{3}k(4k-1) \tag{5}$$

■

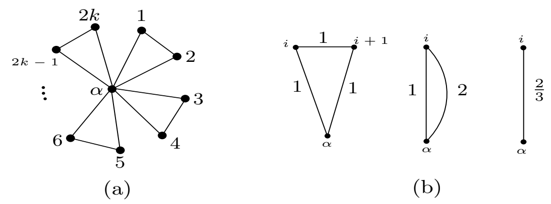


Fig. 3. (a) Labelling of $(\mathcal{F}_1)^k$. (b) $r_{i,\alpha} = 2/3$.

A graph structure of the form $(\mathcal{F}_1)^k \bullet \mathcal{S}_m$, obtained by identifying a petal graph $(\mathcal{F}_1)^k$, and a star graph S_m, is used in the Section 4 for defining a graph process where edges are added to maximize robustness. Following lemma computes the K_f for such a graph.

Lemma 2. Let $G = (\mathcal{F}_1)^k \bullet \mathcal{S}_m$ be a graph with $2k+m$ vertices. Then,

$$K_f(G) = (m-1)^2 + \frac{2}{3}k(5m+4k-6) \tag{6}$$

Proof. Kirchhoff index of a given G can be written as,

$$K_f(G) = K_f((\mathcal{F}_1)^k) + K_f(\mathcal{S}_m) + \sum_{i \in \mathcal{S}_m, j \in (\mathcal{F}_1)^k} r_{i,j} \tag{7}$$

Let α be the central vertex of given G, (i.e., α is the vertex with a degree $2k+m$). Noting that $r_{i,\alpha} = 1$, where i is any of the non-central vertex of $\mathcal{S}_m$. Also, $r_{j,\alpha} = 2/3$, where j is any of the non-central vertex of $(\mathcal{F}_1)^k$. Thus, $r_{i,j} = 5/3$, where $i \in \mathcal{S}_m$ and $j \in (\mathcal{F}_1)^k$. This gives $\sum_{i \in \mathcal{S}_m, j \in (\mathcal{F}_1)^k} r_{i,j} = (m-$

1) $\left[\frac{5}{3}(2k)\right]$. Also, we know that $K_f(\mathcal{S}_m) = (m-1)^2$ (see Ellens et al. (2011) as an example). Using these results along with (4), we get,

$$\begin{aligned} K_f(G) &= \frac{2}{3}k(4k-1) \;+\; (m-1)^2 \;+\; (m-1)\left[\frac{5}{3}(2k)\right] \\ &= (m-1)^2 + \frac{2}{3}k(5m+4k-6) \end{aligned}$$

■

We have also computed the K_f for the following two special graph structures that will be used later. For proofs, readers are referred to Abbas and Egerstedt (2012).

Lemma 3. Let $G = (\mathcal{F}_1)^k \bullet \mathcal{S}_m$. Then the Kirchhoff index of $G' = G \bullet \mathcal{F}_2$ is

$$K_f(G') = (m-1)^2 + \frac{19}{4}m - \frac{3}{4} + \frac{2k}{3}\left(5m + 4k + \frac{21}{4}\right) \tag{8}$$

Lemma 4. Let $G = (\mathcal{F}_1)^k \bullet \mathcal{S}_m$. Then the Kirchhoff index of $G' = G \bullet \mathcal{F}_3$ is

$$K_f(G') = (m-1)^2 + \frac{130}{21}m + \frac{16}{21} + \frac{2k}{3}\left(5m + 4k + \frac{60}{7}\right) \tag{9}$$

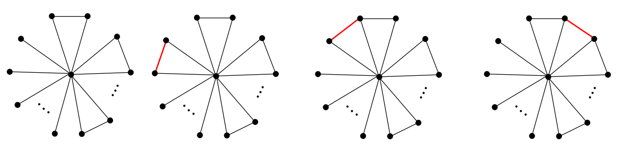
Using these results, we can figure out the best way to add an edge in a graph $G = (\mathcal{F}_1)^k \bullet \mathcal{S}_m$, that will be required to optimally add edges in a graph in a step-wise manner.

Theorem 5. Let $m \geq 2$, and $G = (\mathcal{F}_1)^k \bullet \mathcal{S}_m$. Let H be a graph obtained from G by adding a single edge. Among all such H, $(\mathcal{F}_1)^{k+1} \bullet \mathcal{S}_{m-2}$ has a minimum value of Kirchhoff index.

Proof. Let H be a graph obtained by adding an edge $u \sim v$ between any two non adjacent vertices in $G = (\mathcal{F}_1)^k \bullet \mathcal{S}_m$. Then H is isomorphic to one of the following graphs,

(1) $(\mathcal{F}_1)^{k+1} \bullet \mathcal{S}_{m-2}$
(2) $\mathcal{F}_2 \bullet ((\mathcal{F}_1)^{k-1} \bullet \mathcal{S}_{m-1})$
(3) $\mathcal{F}_3 \bullet ((\mathcal{F}_1)^{k-2} \bullet \mathcal{S}_m)$

This is true as there are only three ways of adding an edge in a given G. An edge can be added between u and v in G where u and v are of degree 1 as shown in the Fig. 4(b). This results in $H = (\mathcal{F}_1)^{k+1} \bullet \mathcal{S}_{m-2}$. When u has a degree 1 and v has degree 2 (equivalently v has degree 1 and u has a degree 2) in an added edge $u \sim v$, we get $H = \mathcal{F}_2 \bullet ((\mathcal{F}_1)^{k-1} \bullet \mathcal{S}_{m-1})$. This is shown in the Fig. 4(c). When both the end vertices of an edge added to G are of degree 2, we get $H = \mathcal{F}_3 \bullet ((\mathcal{F}_1)^{k-2} \bullet \mathcal{S}_m)$, shown in the Fig. 4(d).



$(\mathcal{F}_1)^k \bullet \mathcal{S}_m$ (a) $(\mathcal{F}_1)^{k+1} \bullet \mathcal{S}_{m-2}$ (b) $\mathcal{F}_2 \bullet ((\mathcal{F}_1)^{k-1} \bullet \mathcal{S}_{m-1})$ (c) $\mathcal{F}_3 \bullet ((\mathcal{F}_1)^{k-2} \bullet \mathcal{S}_m)$ (d)

Fig. 4. (a) $(\mathcal{F}_1)^k \bullet \mathcal{S}_m$. Adding an edge to (a) will result into one of the graphs shown in (b), (c) or (d).

Now let $H_1 = (\mathcal{F}_1)^{k+1} \bullet \mathcal{S}_{m-2}$, $H_2 = \mathcal{F}_2 \bullet ((\mathcal{F}_1)^{k-1} \bullet \mathcal{S}_{m-1})$ and $H_3 = \mathcal{F}_3 \bullet ((\mathcal{F}_1)^{k-2} \bullet \mathcal{S}_m)$. Each of these H_1, H_2 and H_3 have same number of edges and are obtained by adding a single edge in G.

Now using (6) and (9), we calculate $K_f(H_3) - K_f(H_1)$ as,

$$K_f(H_3) - K_f(H_1) = \frac{4}{21}(2k+m) > 0 \tag{10}$$

Similary using (6) and (8),

$$K_f(H_2) - K_f(H_1) = \frac{1}{12}(2k+m) > 0 \tag{11}$$

From (10) and (11), we have the following order

$$K_f(H_1) < K_f(H_2) < K_f(H_3)$$

which proves the desired result. ■

4. GRAPH PROCESS FOR STEP-WISE OPTIMAL ADDITION OF EDGES

Addition of an edge in a graph always decreases its K_f (as shown in Ellens et al. (2011)) and hence, increases robustness. But, addition of a certain missing edge may result in a greater decrease in K_f as compared to another edge. Thus, an analysis regarding an optimal addition of edges to minimize the Kirchhoff index is of great significance. As it is discussed in Ellens et al. (2011), the question of determining an optimal edge to add to a graph in order to minimize its K_f is still open. In this section, we provide a systematic way to obtain robust network topologies by optimally adding edges to existing graph structures. We start with a set of nodes without any edge between them, and successively add edges (one at at time) to maximally increase robustness. A notion of Kirchhoff graph process is introduced to characterize such a scheme.

Definition (Kirchhoff Graph Process): A Kirchhoff graph process, $\mathcal{G}$, on n vertices is a sequence of graphs, where $\mathcal{G}_1$ is an edgeless graph on n vertices, and $\mathcal{G}_{i+1}$ is obtained by adding a single edge to $\mathcal{G}_i$ such that $\mathcal{G}_{i+1}$ has a minimum value of Kirchhoff index over all possible choices of $(\mathcal{G}_i + e)$, where $(\mathcal{G}_i + e)$ is a graph obtained by adding a single edge to $\mathcal{G}_i$.

4.1 *Kirchhoff Graph Process from $\mathcal{G}_1$ to $\mathcal{G}_n$*

Note that the number of edges in $\mathcal{G}_i$ is $i-1$. Since there are n nodes, the graph will remain disconnected till $n-1$ step. We know that a graph with n nodes and $n-1$ edges with a minimum K_f is a star graph, $\mathcal{S}_n$ (e.g., see Young et al. (2011)). So, from $i=1$ to $i=n$, edges will be added so as to get $\mathcal{G}_n = \mathcal{S}_n$. Thus,

$$\mathcal{G}_i = \mathcal{S}_i \cup \bar{K}_{n-i} \qquad\qquad i \in \{1, 2, \cdots, n\} \tag{12}$$

where, $\bar{K}_{n-i}$ is an edgeless graph with $n-i$ nodes.

4.2 *Kirchhoff Graph Process from $\mathcal{G}_{n+1}$ to $\mathcal{G}_{n+\lfloor\frac{n-1}{2}\rfloor}$*

Adding an edge to a star graph, $\mathcal{S}_n$ will always result in a $\mathcal{F}_1 \bullet \mathcal{S}_{n-2}$ graph. Thus, $\mathcal{G}_{n+1} = \mathcal{F}_1 \bullet \mathcal{S}_{n-2}$. The optimal way to add an edge in subsequent steps is to connect two non adjacent vertices having a degree 1 as shown in the Fig. 5(b). In fact, Theorem 5 and Lemma 2 provides an optimal way to add an edge in $(\mathcal{F}_1^k) \bullet \mathcal{S}_m$. Using these results, we get instances of the Kirchhoff graph process $\mathcal{G}_i$ for $i \in \{n+1, \cdots, \lfloor\frac{n-1}{2}\rfloor\}$ as,

$$\mathcal{G}_{n+i} = (\mathcal{F}_1)^i \bullet \mathcal{S}_{n-2i} \qquad\qquad i \in \{1, 2, \cdots, \lfloor\frac{n-1}{2}\rfloor\} \tag{13}$$

For a simpler case, let n be an odd number. Then, for $i = \left(\frac{n-1}{2}\right)$, $\mathcal{G}_{n+i}$ is a petal graph, $(\mathcal{F}_1)^{\frac{n-1}{2}} \bullet \mathcal{S}_1 = (\mathcal{F}_1)^{\frac{n-1}{2}}$.

4.3 *Adding edges to a Petal Graph*

Adding an edge to a petal graph of the form $(\mathcal{F}_1)^k$ always results in a graph $(\mathcal{F}_1)^{k-2} \bullet \mathcal{F}_3$. Thus, in a Kirchhoff graph process,

$$\mathcal{G}_i = (\mathcal{F}_1)^{(\frac{n-1}{2}-2)} \bullet \mathcal{F}_3 \qquad\qquad i = n + \frac{n-1}{2} + 1 \tag{14}$$

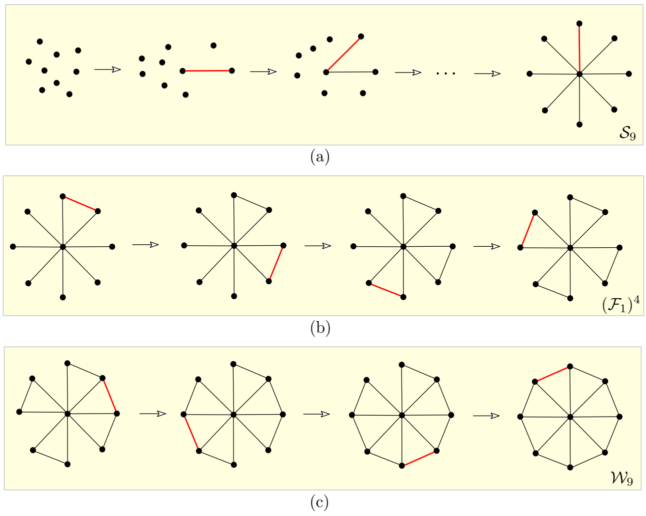


Fig. 5. A Kirchhoff graph process for $n = 9$ nodes.

Also, it can be shown that (see Abbas and Egerstedt (2012)), if a graph is of the form $(\mathcal{F}_1)^k \bullet (\mathcal{F}_3)^\ell$, then optimal addition of a single edge minimizing the Kirchhoff index yields a graph $(\mathcal{F}_1)^{k-2} \bullet (\mathcal{F}_3)^{\ell+1}$. This results provides a way of adding edges to instances of a graph process $\mathcal{G}_i$ for $i > n + \left(\frac{n-1}{2}\right)$. An example is also shown in the Fig. 5(c). Further analysis of this process shows that edges are being added in a specific pattern. From a star graph at $\mathcal{G}_n = \mathcal{S}_n$, edges are added to increase the number of 1-petals (i.e., $\mathcal{F}_1$) in the intermediate steps of the Kirchhoff graph process until a petal graph, where every petal is a 1-petal is obtained. Similarly, from a 1-petal graph at $\mathcal{G}_{n+\frac{n-1}{2}} = (\mathcal{F}_1)^{\frac{n-1}{2}}$, edges are added to increase the number of 3-petals (i.e. $\mathcal{F}_3$) by connecting two 1-petals. This continues till a petal graph, where every petal is a 3-petal is obtained. In the next steps, edges are added to 3-petal graph such that at each step two 3-petals are combined to give 7-petal. This continues until a wheel graph $\mathcal{W}_n$ is obtained at the $2n - 1$ step of the Kirchhoff graph process, that is,

$$\mathcal{G}_{2n-1} = \mathcal{W}_n \tag{15}$$

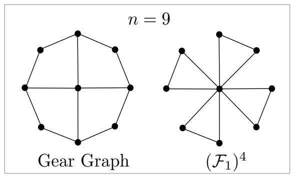
It is to be noted here that at each step of the Kirchhoff graph process, an edge is added optimally to maximize robustness property of a graph. An example for $n = 9$ vertices is shown in the Fig. 5.

4.4 *Step-wise Optimal Graph vs. Globally Optimal Graph*

Consider a graph with n vertices for some odd integer n, containing $(n - 1) + \left(\frac{n-1}{2}\right)$ edges, and obtained through a Kirchhoff graph process. From, (13), we know that it is a graph of the form $(\mathcal{F}_1)^{\frac{n-1}{2}}$. Its Kirchhoff index can be computed using the Lemma 1 for any n. A gear graph[2] with n verices and $(\mathcal{F}_1)^{\frac{n-1}{2}}$ has the same number of vertices and edges. It is observed that for many different values of n, a gear graph with n vertices has a smaller K_f than $(\mathcal{F}_1)^{\frac{n-1}{2}}$. This implies that although $(\mathcal{F}_1)^{\frac{n-1}{2}}$ is obtained by optimally adding edges in a step-wise manner, still it is not a graph with a minimum K_f for a given number of nodes and edges. A comparison of K_f values for a gear graph and a petal graph with the same number of nodes and edges is shown in Fig. 6(b). Thus, optimal step-wise addition of edges does not necessarily give a globally optimum graph, i.e. a graph with a minimum K_f for a given number of nodes and edges. We can state it as a following Proposition.

[2] A gear graph with $2m + 1$ vertices is obtained from a wheel graph $\mathcal{W}_m$, by adding a vertex between each pair of adjacent vertices on the outer cycle of $\mathcal{W}_m$ (see Fig. 6).

Proposition 6. A graph G with $\mathcal{E}$ number of edges, obtained through a Kirchhoff graph process by optimally adding a single edge at each step of the process to minimize K_f, does not necessarily give a globally optimum graph having a minimum K_f among all graphs with n nodes and $\mathcal{E}$ edges.



(a)

n	K_f(Gear graph)	K_f $((\mathcal{F}_1)^{\frac{n-1}{2}})$
9	34.5	40
11	57.11	63.33
13	85.67	92
15	120.08	136
17	160.31	165.33
19	206.32	210

(b)

Fig. 6. (a) A gear graph with 9 nodes and a petal graph, $(\mathcal{F}_1)^4$. (b) Comparison of K_f of gear graph and $(\mathcal{F}_1)^k$ with same number of vertices, n, and edges.

5. SYMMETRY OF NETWORKS AND ROBUSTNESS

Symmetric network topologies are more robust and have a smaller Kirchhoff index (see Ellens et al. (2011) and Wu et al. (2011) as examples). In fact, for a given number of nodes and diameter, a special graph known as a *clique chain* (see Ellens et al. (2011)), which is a symmetric structure, has a minimum value of effective resistance and therefore, maximum robustness. Similarly for a given number of nodes, a complete graph which is also symmetric, has a maximum robustness. A relationship between symmetry and robustness can also be seen in the Kirchhoff graph process discussed in Section 4. At each step of the process, edges are added so as to preserve the symmetry of the overall graph. Thus, symmetry of a graph has a far reaching impact on its robustness properties.

Here, we show an optimal (in the sense of minimizing the K_f) way to attach a path graph to an arbitrary graph G. Again it is observed that symmetry of a graph plays an important role in minimizing K_f. Let G be any graph with j number of nodes, where $j > 1$. A *vine graph* is obtained from a graph G by attaching two separate paths with i and p number of nodes to G through two of its nodes. Let a path $\mathcal{P}_i$ be connected to G through node 1 and a path, $\mathcal{P}_p$, through node j of G. A vine graph, denoted by $G_{\{i,p\}}$ is shown in the Fig. 7 (a). In a vine graph, paths of i and p nodes may be connected to G through the same vertex, as shown in the Fig. 7 (b). In Young et al. (2011), it is shown that if G is a *tree*, T, and paths $\mathcal{P}_i$ and $\mathcal{P}_p$, where $1 \leq i \leq p$, are connected to T through a common vertex, then,

$$K_f\left(T_{\{i,p\}}\right) < K_f\left(T_{\{i-1,p+1\}}\right) \tag{16}$$

Here, we generalize this result and show that (16) holds even if trees are replaced with any graphs. In fact, we provide a necessary and sufficient condition for $K_f\left(G_{\{i,p\}}\right) < K_f\left(G_{\{i-1,p+1\}}\right)$ to be true even when paths with i and p number of nodes are connected to G through two different vertices, say 1 and j respectively. For a detailed proof of the Theorem 7, readers are referred to Abbas and Egerstedt (2012).

Theorem 7. Let G be a graph with $j > 1$ nodes. Let a path $\mathcal{P}_i$ be connected to G through a node, say 1 of G. Another path, $\mathcal{P}_p$ be connected to G through a node, say j of G, to get a vine graph $G_{\{i,p\}}$, where, $1 \leq i \leq p$. Then

$$K_f\left(G_{\{i,p\}}\right) < K_f\left(G_{\{i-1,p+1\}}\right) \tag{17}$$

if and only if

$$(p+1-i)(j-1-r_{1,j}) > \sum_{s=1}^{j} r_{1,s} - \sum_{s=1}^{j} r_{s,j} \tag{18}$$

Proof. Without loss of generality, let us label the vertices in $G_{\{i,p\}}$ as shown in the Fig. 7. Then, we can write the K_f of $G_{\{i,p\}}$ as follows,

$$K_f(G_{\{i,p\}}) = \sum_{1 \le s < t \le (j+i+p)} r_{s,t}$$
$$= \sum_{s=1}^{j} \sum_{t>s}^{j} r_{s,t} \; + \; \underbrace{\sum_{s=1}^{j} \sum_{t=(j+1)}^{(j+i+p)} r_{s,t}}_{A} + \underbrace{\sum_{s=(j+1)}^{(j+i+p)} \sum_{t>s}^{(j+i+p)} r_{s,t}}_{B} \tag{19}$$

Let us compute the second term in (19).

$$\begin{aligned} A &= \sum_{t=(j+1)}^{(j+i+p)} r_{1,t} \; + \; \sum_{t=(j+1)}^{(j+i+p)} r_{2,t} \; + \cdots \sum_{t=(j+1)}^{(j+i+p)} r_{j,t} \\ &= \frac{ij}{2}(1+i) \; + \; \frac{jp}{2}(1+p) \; + \; p\sum_{s=1}^{j} r_{s,j} \; + \; i\sum_{s=1}^{j} r_{1,s} \end{aligned} \tag{20}$$

Now, computing the third term in (19), B, gives,

$$B = \frac{1}{6}\left[i(i^2-1) + p(p^2-1)\right] + \frac{ip}{2}\left[p+i+2+2r_{1,j}\right] \tag{21}$$

Kirchhoff index for $G_{\{i-1,p+1\}}$ can also be written in an exactly similar way as in (19), with the corresponding A' and B' terms are computed as,

$$\begin{aligned} A' &= \frac{ij}{2}(i-1) \; + \; \frac{j}{2}(p+1)(p+2) + \; (p+1)\sum_{s=1}^{j} r_{s,j} \\ &+ \; (i-1)\sum_{s=1}^{j} r_{1,s} \end{aligned} \tag{22}$$

$$\begin{aligned} B' &= \frac{1}{6}\left[i(i^2-3i+2) + p(p^2+3p+2)\right] \\ &+ \frac{1}{2}(i-1)(p+1)\left(p+i+2+2r_{1,j}\right) \end{aligned} \tag{23}$$

Inserting (20) and (21) into (19) gives, $K_f(G_{\{i,p\}})$ and inserting (22) and (23) gives $K_f(G_{\{i-1,p+1\}})$. Now calculating $K_f(G_{\{i-1,p+1\}}) - K_f(G_{\{i,p\}})$ gives the following after some simplifications,

$$\begin{aligned} K_f(G_{\{i-1,p+1\}}) - K_f(G_{\{i,p\}}) &= \sum_{s=1}^{j} r_{s,j} - \sum_{s=1}^{j} r_{1,s} \\ &+ (p+1-i)(j-1-r_{1,j}) \end{aligned} \tag{24}$$

The required result directly follows from (24). ■

A special case of the above theorem is when $\mathcal{P}_i$ and $\mathcal{P}_p$ are connected to G through the same vertex, say 1 (as shown in the Fig. 7(b)). The condition in (18) is then, always satisfied as long as $1 \le i \le p$. This is true as 1 and j in (18) correspond to the same vertex here and so, $\sum_{s=1}^{j} r_{1,s} = \sum_{s=1}^{j} r_{s,j}$, and $r_{1,j} = 0$. Also, $(j-1) > 0$, as long as G has at least two nodes. A proof of the following Theorem can be found in Abbas and Egerstedt (2012).

Theorem 8. Let G be a graph with at least two nodes. Let two paths with i and p number of nodes respectively, are connected to G through the same vertex of G to get $G_{\{i,p\}}$. Then,

$$K_f(G_{\{i,p\}}) < K_f(G_{\{i-1,p+1\}}) \tag{25}$$

Here, $1 \le i \le p$.

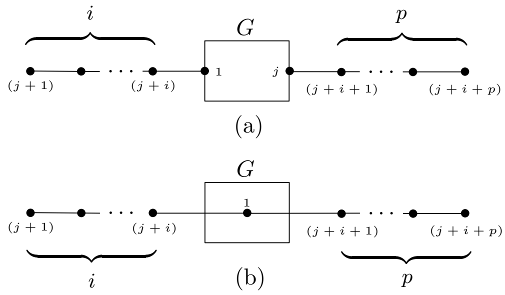


Fig. 7. (a) Paths $\mathcal{P}_i$ and $\mathcal{P}_p$ are connected to G through nodes 1 and j of G, respectively. In (b), both paths $\mathcal{P}_i$ and $\mathcal{P}_p$ are connected through a same vertex, 1.

It is to be mentioned here that the symmetry of an underlying graph also plays an important role in determining some other properties of networked systems with agents implementing a linear consensus protocol. One such noticeable property is the controllability of such systems under a leader-follower setting, where external inputs are injected through so called leader nodes. Structures that are symmetric about a leader exhibit poor controllability properties (see Mesbahi and Egerstedt (2010) as an example). For example, a complete graph (most robust network for a given number of nodes) is least controllable. Thus, we can say that from a network topology perspective, controllability and robustness properties are in conflict with each other. Improving one by reconfiguring the underlying graph structure may deteriorate the other one. A precise relationship between these two properties in terms of the graph structure is an interesting research direction.

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