ONTARIO MATH CIRCLES
ANNUAL ARML TEAM SELECTION TEST
2017

Instructions

1. Do not open this competition package until you are told to do so.
2. This competition contains 40 problems to be answered in 120 minutes. This is open to all high school and middle school students.
3. Print your name clearly on your answer sheet. Phone number and e-mail addresses are requested to contact top finishers about being on the Ontario Math Circles ARML team. If you do not wish to be consider for this team, you may leave those blank. Please print these, especially e-mail address, legibly.
4. Print clearly and legibly on the answer package. You must leave all answers in exact form. For example, π would be correct while 3.1415 would not. You must simplify your answers as much as possible. For example, $\frac{6}{4}$ must be simplified to $\frac{3}{2}$ and all denominators must be rationalized. Perfect squares must be removed from radicals. For example, $\sqrt{99}$ must be written as $3\sqrt{11}$. Trig functions of standard arguments must be evaluated. Frequently, several equivalent expressions will be considered correct. For example, $\frac{3}{2}$, $1\frac{1}{2}$, and 1.5 will all be considered correct.
5. Each correct answer is worth 1 point. A blank and an incorrect answer are both worth 0 points. There are no fractional points.
6. Only pencils, erasers, and pens are allowed. **Electronic devices and calculators are not allowed.** Everything else must be approved by the proctor.
7. You may keep your question sheet and scratch work.
8. The full results of this competition will be posted on the Toronto Math Circles' website. It will only display your name, school, grade, and score.
9. The prize for top scorers will be a numerical amount to be deducted from one of their future ARML trip fees. This amount is not transferable nor redeemable for cash.
10. ***Do not discuss the problems or solutions from this contest for the next 7 days.***
11. GOOD LUCK!

1. Determine the value of $\frac{6^2 \times 35^2}{14^2 \times 15^2}$.

2. Determine the number of ways to place three As, three Bs, and three Cs in a 3×3 grid such that each row and column contain one of each letter.

3. Determine all values of x that satisfy the equation

$$x^2 - 4034x + 4068289 = 0$$

4. Let A, B, and C be three non-collinear points and M be a point on AB. Let P and Q be two points on the plane such that PM is the angle bisector of $\angle CMA$ and QM is the angle bisector of $\angle CMB$. Determine the measure of $\angle PMQ$.

5. Determine all real values of m such that

$$(m^2 - 1)x^2 + 1 = 2(m - 1)x$$

   has at least one real solution.

6. An ant is traveling from the point $(0, 0)$ to $(4, 5)$. In each move, the ant can only move one unit to the right or one unit up. Determine the probability of the ant passing through $(2, 3)$.

7. In a party, every person must shake hands with every other person exactly once. There are currently 10 people at the party, who have already shaken hands. 10 more people will arrive one after another, how many new handshakes will occur?

8. 46 is a base 7 number, determine the equivalent base 3 number.

9. Determine the number of ordered positive integral triplets (a, b, c) that satisfy the following system of equations

$$\begin{cases} abc = 2016 \\ (a-1)(b-1)(c-1) = 1573 \end{cases}$$

10. A rectangular prism has integral side lengths and has a volume of 9. Determine the sum of all possible surface areas of this rectangular prism.

11. Determine the sum of all real solutions to the equation $\sin x = x^{2017}$.

12. Let x_1 be a solution to the equation

$$2x + 2^x = 8$$

   and let x_2 be a solution to the equation

$$2x + 2\log_2(x - 1) = 4$$

   Determine all possible values of $x_1 + x_2$.

13. Given that $\frac{\sin(x+y)}{\sin(x-y)} = 2017$, determine the value of $\frac{\tan x}{\tan y}$.

14. Let $\frac{a}{b+c} = \frac{b}{a+c} = \frac{c}{a+b}$ where a, b, c are real numbers. Determine the sum of all possible values of $\frac{a}{b+c}$.

15. Find the nearest integer to $(17 - 12\sqrt{2})^{-0.5}$.

16. Let a, b, and c be positive integer numbers such that $c \geq a$ and $a + c = b$. Determine the largest possible value of the smaller root of $ax^2 + bx + c = 0$.

17. Let P be the point $(3, 2)$. If A is a point on the line $y = x$ and B is a point on the $x-$axis such that the perimeter of $\triangle ABP$ is minimized, determine the midpoint of the line segment AB.

18. Let S be the set of all four digits numbers such that each digit is either $1, 2$, or 3. Determine the number of ways to select three distinct numbers from S such that, for $i = 1, 2, 3, 4$, the i^{th} digit in all three numbers are either all equal or all unique. For example,

$$1111, 2221, 3331$$

is one way to select the three numbers.

19. Let $x > 1$ and

$$(\log_x 128)(\log_{128} 16) = \sqrt[3]{72}$$

determine the value of $\log_{\frac{2}{x}} 256$.

20. The current time on a clock face is 7AM. Determine the next time, rounded to the nearest second, such that the minute hand and the hour hand are on top of each other. Express your answer in the format of Hours:Minutes:Seconds.

21. Determine the sum of the prime factors of 14541.

22. Let x and y be real numbers such that $2x + y = 1$ and $|y| \leq 1$. Determine the minimum possible value of $2x^2 + 16x + 3y^2$.

23. The following two curves

$$\begin{cases} y = (x-1)^2 \\ x = (y-1)^2 \end{cases}$$

intersects at four distinct points. Determine the area of the quadrilateral formed by these four points.

24. Given that the following equation

$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0$$

has exactly two solutions, α and β. Determine the value of $\alpha\beta$.

25. Let $ABCD$ be a rectangle with $\overline{AD} = 12$ and $\overline{AB} = 5$. Let P be the point on the line segment $\overline{AD}$ such that $\left|\overline{AP}\right| = \sqrt{2}$. Denote h_1 to be the distance from P to AC and h_2 to be the distance from P to BD. Determine the value of $h_1 + h_2$.

26. Determine the largest integer N such that $a^{13} - a$ is divisible by N for every integer $0 \leq a \leq N - 1$.

27. Let a, b, c be complex numbers such that

$$|a| = |b| = |c| = 2a + 2b + 2c = abc = 1$$

Determine one possible triplet (a, b, c).

28. Let a be a solution to the equation $2x^2 - 3x - 1 = 0$ and let b be a solution to the equation $x^2 + 3x - 2 = 0$. If $ab \neq 1$, determine the value of $\frac{ab+2017a+1}{b}$.

29. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where a_i are non-negative integers for $i = 0, 1, \ldots, n$. If $f(1) = 21$ and $f(25) = 78357$. Determine $f(5)$.

30. Let $f(x) = x^2 + 32x + 240$. Determine the product of all real solutions to the equation

$$f(f(f(x))) = 6545$$

31. Determine the number of terms in the expansion of $(x + y + z)^{101}$ where the coefficient is divisible by 101.

32. Let $X = (1+i)^{2016}$ be a complex number. Determine the number of digits of

$$\left\lfloor \frac{1}{2} \left| (X(1-i)) + \bar{X}(1+i) \right| \right\rfloor$$

33. Let a, b, c be the lengths of the three sides of $\triangle ABC$ such that $a > b > c$, $2b > a + c$, and b is a positive integer. If $a^2 + b^2 + c^2 = 84$, determine the value of b.

34. Determine the sum of all integers k such that $0 \le k \le 72$ and

$$\left\lfloor \sqrt{\frac{1}{k!} \prod_{n=1}^{72} n!} \right\rfloor - \sqrt{\frac{1}{k!} \prod_{n=1}^{72} n!} = 0$$

35. Determine the number of ordered integral triplets (a, b, c) with $0 \le a, b, c \le 100$ such that

$$a^3(b-c) + b^3(c-a) + c^3(a-b) = 0$$

36. Let n be a positive integer. Consider the family of 46 curves

$$ky = x^2 + k^2$$

where $k = 45, 44, 43, \ldots, 2, 1, 0$. Determine the number of regions into which these 46 curves divide the plane.

37. Compute the number of permutations of $x_1, x_2, \ldots, x_{10}$ of the integers $-3, -2, -1, \ldots, 6$ that satisfy the chain of inequality

$$x_1 x_2 \le x_2 x_3 \le \cdots \le x_9 x_{10}$$

38. Let ABC be an isosceles triangle with $\angle ABC = \angle ACB$. Let P be a point inside $\triangle ABC$ such that $\angle BCP = 30°$, $\angle APB = 150°$, and $\angle CAP = 39°$. Determine the value of $\angle BAP$.

39. Determine the smallest positive integer n such that for any set S of n real numbers, there exists a and b in S such that

$$0 < \frac{a-b}{1+ab} \le 1$$

40. Determine all positive real numbers x such that

$$\lfloor 2x \rfloor, \lfloor x^2 \rfloor, \lfloor x^4 \rfloor$$

are the three side lengths of a non-degenerate triangle.