Math 3331 – ODE's

First order

$$\frac{dy}{dx} = F(x,y)$$

Techniques

(1) Separable $\quad \frac{dy}{dx} = f(x)g(y)$

so $\quad \frac{dy}{g(y)} = f(x)dx$

$$\int \frac{dy}{g(y)} = \int f(x)dx + c$$

Ex $\quad \frac{dy}{dx} = \frac{1}{xy^2}$

$$y^2 dy = \frac{dx}{x} \Rightarrow \frac{y^3}{3} = \ln|x| + c$$

Could solve for y if we want.

(2) <u>Linear</u>

These ODEs are of the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

if $p = 0$ or $q = 0$ the ODE is separable

so we assume $p \neq 0, q \neq 0$

Before trying to solve this let's review the product rule

$$\frac{d}{dx} xy = x\frac{dy}{dx} + y$$

$$\frac{d}{dx}(e^x y) = e^x \frac{dy}{dx} + e^x y$$

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{d}{dx} x^{-2} y = x^{-2}\frac{dy}{dx} - 2x^{-3} y$$

$$= \frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3}$$

some of our ODE is starting to appear

Consider $$\frac{dy}{dx} + \frac{y}{x} = 2$$

Here $p(x) = \frac{1}{x}$ $q(x) = 2$

if we multiply by x

$$\underbrace{x\frac{dy}{dx} + y}_{} = 2x$$

$$\frac{d}{dx}(xy) = 2x$$ ← why is this nice
it separates

$$d(xy) = 2x\,dx$$

$$\int d(xy) = \int 2x\,dx + c$$

$$xy = x^2 + c \Rightarrow y = x + c/x$$ Sol^y

consider $$\frac{dy}{dx} - \frac{2y}{x} = x$$

multiply by $\frac{1}{x^2}$ $$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = \frac{x}{x^2}$$

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x} \quad \leftarrow \text{separates}$$

$$d\left(y/x^2\right) = \frac{1}{x}\,dx$$

$$y/x^2 = \ln|x| + c \Rightarrow y = x^2(\ln|x| + c)$$

so how did we know what to mult by?

given $\quad \frac{dy}{dx} + p(x)y = q(x)$

μ - integrating factor

$$\underbrace{\mu\frac{dy}{dx} + p(x)y\mu}_{\frac{d}{dx}(\mu y)} = \mu q$$

Expand this & equate

$$\cancel{\mu\frac{dy}{dx}} + \frac{d\mu}{dx}y = \cancel{\mu\frac{dy}{dx}} + p(x)y\mu \Rightarrow \frac{d\mu}{dx} = p(x)\mu$$

separable

so $\dfrac{d\mu}{\mu} = p(x)\,dx$

$$\ln|\mu| = \int p(x)\,dx$$

$$\mu = e^{\int p(x)\,dx}$$

<u>check</u>

$$\frac{dy}{dx} + \frac{y}{x} = 2$$

$$p(x) = \frac{1}{x} \qquad \mu = e^{\int \frac{dx}{x}} = e^{\ln|x|} = x$$

$$\frac{dy}{dx} - \frac{2y}{x} = x$$

$$p(x) = -\frac{2}{x} \qquad \mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

Ex $\cos x \frac{dy}{dx} - 2\sin x\, y = 1 \qquad y(0) = 4$

S.F. $\frac{dy}{dx} - \frac{2\sin x}{\cos x} y = \frac{1}{\cos x}$

$\mu = e^{2\int \frac{-\sin x}{\cos x} dx} = e^{2\ln|\cos x|} = \cos^2 x$

so $\cos^2 x \frac{dy}{dx} - 2\sin x \cos x\, y = \cos x$

$\frac{d}{dx} \cos^2 x\, y = \cos x$

$\cos^2 x\, y = \sin x + c \qquad y(0) = 4$

$\cos^2(0)\, 4 = \sin 0 + c \Rightarrow c = 4$

$\cos^2 x\, y = \sin x + 4$

$y = \frac{\sin x + 4}{\cos^2 x}$