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Research Article

Certain Aspects of Normal Classes of Hilbert Space Operators

N. B. Okelo*

School of Mathematics and Actuarial Science,
Jaramogi Oginga Odinga University of Science and Technology,
P. O. Box 210-40601, Bondo-Kenya.

*Corresponding author's e-mail: bnyaare@yahoo.com

Abstract

Let T be a Quasi - * - class A normal operator on a complex Hilbert space H. In this paper, we prove that if E is the Riesz idempotent for a non-zero isolated point λ of the spectrum of $T \in B(H)$ of Quasi - * - class A normal operator, then E is self-adjoint and $EH = \ker(T-\lambda) = \ker(T-\lambda)^*$. We will also prove a necessary and sufficient condition for $T \otimes S$ to be quasi - * - class A *normal* where T and S are both non-zero operators.

Keywords: Paranormal operators; Weyl's theorem; * - class A normal operators; Quasi - * - class A normal operators.

Introduction

Studies on Hilbert space operators has been carried out over a period of time by several authors [1]. Let $B(H)$ denote the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space H. For a positive operators A and B, we write $A > B$ if $A - B > 0$. If A and B are invertible [2] and positive operators, it is well known that $A > B$ implies that $\log A > \log B$ [3]. However from [4], $\log A > \log B$ does not necessarily imply $A > B$. A result due to [5] states that for invertible positive operators A and B, $\log A > \log B$ if and only if $A^r > (A^{r2} B^r A^{r2})^{1/2}$ for all $r > 0$ [6]. For an operator T, let $U|T|$ denote the polar decomposition of T, where U is a partially isometric operator, $|T|$ is a positive square root of T^*T and $\ker(T) = \ker(U) = \ker(|T|)$, where $\ker(T)$ denotes the kernel of operator T [7]. An operator $T \in B(H)$ is positive, $T > 0$, if $(Tx, x) > 0$, for all $x \in H$ and posinormal if there exists a positive λ such that $TT^* = T^*\lambda T$. Here λ is called interrupter of T [8]. In other words, an operator T is called posinormal if $TT^* < c^2T^*T$, where T^* is the adjoint of T and $c > 0$ [9]. An operator T is said to be herminormal if T is hyponormal and T^*T commutes with TT^*. An operator T is said to be p - posinormal if $(TT^*)p < c^2(T^*T)p$ for some $c > 0$ [10]. It is clear that p - posinormal is posinormal. An operator T is said to be p - hyponormal, for $p \in (0, 1)$, if $(T^*T)p > (TT^*)p$. In [11], they have characterized class A operator as follows. An operator T belongs to class A if and only if $(T^*|T|T)^{1/2} \geq T^*T$. An operator T is said to be paranormal if $||T^2x|| \geq ||Tx||^2$ and $*$ - paranormal if $||T^2x|| \geq ||T^*x||^2$ for all unit vector $x \in H$ [12].

Recently, authors in [13] have considered the new class of operators: An operator $T \in B(H)$ belongs to $*$ - class A *normal* if $|T^2| \geq |T^*|^2$. The authors of [14] have extended $*$ - class A *normal* operators to quasi - $*$ - class A *normal* operators. An operator $T \in B(H)$ is said to be quasi - $*$ - class A *normal* if $T^*|T^2|T \geq T^*|T^*|^2T$ and quasi - $*$ - paranormal if $||T^*Tx||^2 \leq ||T^3x||||Tx||$, for all $x \in H$ [15]. An operator T is said to be Quasi - $*$ - class A *normal* operator [16] on a complex Hilbert space H if $T^*(|T^2| - |T^*|^2)T \geq 0$.

As a further generalization, [17] has introduced the class of k - quasi - $*$ - class A *normal* operators. An operator T is said to be k - quasi - $*$ - class A *normal* operator on a complex Hilbert space H if $T^*(|T^2| - |T^{*2})T \geq 0$ where k is a natural number. An operator T is called normal if $T^*T = TT^*$ and (p, k) - quasihyponormal if $T^{*k}((T^*T)^p - (TT^*)^p)T \geq 0$ $(0 < p \leq 1, k \in \mathbb{N})$. The authors in [18-23] introduced p - hyponormal, p -

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quasihyponormal and k - quasihyponormal operators, respectively. The following classification has been done on these operarors [24, 25, 26]: p - hyponormal $\subset p$ - posinormal $\subset (p, k)$ - quasiposinormal, p - hyponormal $\subset p$ -quasihyponormal $\subset (p, k)$ - quasihyponormal $\subset (p, k)$ – quasiposinormal and hyponormal $\subset k$ - quasihyponormal $\subset (p, k)$ - quasihyponormal $\subset (p, k)$ – quasiposinormal for a positive integer k and a positive number $0 < p \leq 1$. If $T \in B(H)$, we shall write $N(T)$ and $R(T)$ for the null space and the range of T, respectively [27].

Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T, respectively. An operator T is called Fredholm [28] if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim H/R(T) < \infty$. Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl $\sigma W(T)$ are defined by $\sigma_e(T) = \{\lambda \in C : T - \lambda$ is not Fredholm$\}$and$\sigma W(T) = \{\lambda \in C : T - \lambda$ is not Weyl$\}$,respectively. It is known [29,30] that $\sigma_e(T) \subset \sigma W(T) \subset \sigma_e(T) \cup$ acc $\sigma(T)$ where we write acc K for the set of all accumulation points of $K \subset C$. If we write iso $K = K \backslash$ acc K, then we let $\pi 00(T) = \{\lambda \in$ iso $\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$. We say that Weyl's theorem holds for T if $\sigma(T) \backslash \sigma W(T) = \pi 00(T)$.

Let $\sigma_p(T)$ denotes the point spectrum of T, i.e., the set of its eigenvalues. Let $\sigma j_p(T)$ denotes the joint point spectrum of T. We note that $\lambda \in \sigma j_p(T)$ if and only if there exists a non-zero vector x such that $Tx = \lambda x$, $T^* x = \lambda x$. It is evident that $\sigma j_p(T) \subset \sigma_p(T)$. It is well known that, if T is normal, then $\sigma j_p(T) = \sigma_p(T)$. If $T = U|T|$ is the polar decomposition of T and $\lambda = |\lambda| e^{i\theta}$ be the complex number, $|\lambda| > 0$, $|e^{i\theta}| = 1$. Then $\lambda \in \sigma j_p(T)$ if and only if there exist a non-zero vector x such that $Ux = e^{i\theta}$, $|T|x = |\lambda|x$. Let $\sigma_{ap}(T)$ denotes the approximate point spectrum of T, i.e., the set of all complex numbers λ which satisfy the following condition: there exists a sequence $\{x_n\}$ of unit vectors in H such that $\lim_n (T - \lambda)x_n = 0$. It is evident that $\sigma_p(T) \subset \sigma_{ap}(T)$. It is evident that $\sigma_{jap}(T) \subset \sigma_{ap}(T)$, for all $T \in B(H)$. It is well known [5] that, for a normal operator T, $\sigma_{jap}(T) = \sigma_{ap}(T) = \sigma(T)$. An operator $T \in B(H)$ is said to have the single-valued extension property (or SVEP) if for every open subset G of C and any analytic function $f : G \rightarrow H$ such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. An operator $T \in B(H)$ is said to have Bishop's property (β) if for every open subset G of C and every sequence $f_n : G \rightarrow H$ of H - valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in B(H)$ is said to have Dunford's property (C) if $H\boldsymbol{T}(F)$ is closed for each closed subset F of C.

It is well known [7, 9] that Bishop's property $(\beta) \Rightarrow$ Dunford's property $(C) \Rightarrow$ SVEP. Let $T \in B(H)$ and let A_0 be an isolated point of $u(T)$. Then there exists a positive number $r > 0$ such that $\{A \in C: A - A_0 \leq r\}$ $flu(T) = \{A_0\}$. Let γ be the boundary of $\{A \in C: A - A_0 \leq r\}$. In general, it is well known that the Riesz idempotent E is not an orthogonal projection and a necessary and sufficient condition for E to be orthogonal is that E is self-adjoint.

In [15], the author showed that if T satisfies the growth condition G_1, then E is self-adjoint and $E(H) = \ker(T - A_0)$. Recently, [11] and [18] obtained Stampfli's result for quasi - class A *normal* operators and paranormal operators. In general even if T is a paranormal operator, the Riesz idempotent E of T with respect to A_0 is not necessarily self - adjoint. In this study we show that if E is the Riesz idempotent for a nonzero isolated point A_0 of the spectrum of a quasi - $*$ - class A *normal* operator T, then E is self - adjoint and $EH = \ker(T - A_0) = \ker(T^* - A_0)$.

Materials and methods

Lemma 2.1.

([12, Theorem 2.2, Theorem 2.3]) (1) Let T E B(H) be quasi - $*$ - class A operator and T does not have a dense range, then **i**f T is an quasi - $*$ - class A operator and M is its invariant subspace, then the restriction $T_{\mathbf{M}}$ of T to M is also an quasi - $*$ - class A operator.

Lemma 2.2.

[12, Theorem 2.4] Let $T \in B(H)$ is an quasi - $*$ - class A operator. If $A = 0$ and $(T - A)x = 0$ for some x E H, then $(T - A)^* x = 0$.

Lemma 2.3.

Let T ∈ B(H) is an quasi - * - class A operator. Then T is isoloid.

Proof.

Let T∈B(H) is an quasi - * - class A operator with representation given in Lemma 2.1. Let z be an isolated point in σ(T). Since σ(T) = σ(T1) U {0}, z is an isolated point in σ(T1) or z = 0. If z is an isolated point in σ(T1), then z ∈ σ_p(T1). Assume that z = 0 and z is not in σ(T1). This completes the proof.

Theorem 2.4.

Let A ∈ B(H) is an quasi - * - class A normal operator and let A be a non-zero isolated point of σ(A). Let D**A** denote the closed disk that centered at A such that D**A** fl σ(A) = {A}. Then the Riesz idempotent E is self adjoint.

Proof.

If A is quasi - * - class A normal operator, then A is an eigenvalue of A and EH = ker(A − A)* by Lemma 2.3. Since ker(A – A*) ⊂ ker(A − A)* by Lemma 2.2, it suffices to show that ker(A − A)* ⊂ ker(A − A). Since ker(A − A) is a reducing subspace of A by Lemma 2.2 and the restriction of a quasi - * - class A normal operator to its reducing subspaces is also a quasi - * - class A normal operator by Lemma 2.1, hence A can be written as follows: A = A ⊕ A1 on H = ker(A − A) ⊕ (ker(A − A))', where A1 is *-class A normal with ker(A1 − A) = {0}. Since A E σ(A) = {A} U σ(A1) is isolated, the only two cases occur, one is A ∈ σ(A1) and the other is that A is an isolated point of σ(A1) and this contradicts the fact that ker(A1 −A) = {0}. Since A1 is invertible as an operator on (ker(A−A))', ker(A−A) = ker(A−A)*. Next, we show that E is self-adjoint. Since EH=ker(A−A)=ker(A−A)*,we have ((z − A)*)$^{-1}$E = (z − λ)$^{-1}$E. This completes the proof.

Results and discussions

The tensor products $T \otimes S$ preserves many properties of $T, S \in B(H)$, but by no means all of them. Thus, whereas the normaloid property is invariant under tensor products; again, whereas $T \otimes S$ is normal if and only if T and S are normal [10, 16], there exist paranormal operators T and S such that $T \otimes S$ is not paranormal [4]. It is shown in [11] that $T \otimes S$ is quasi-class A if and only if S, T are quasi-class A operators. In the following theorem we will prove a necessary and sufficient condition for $T \otimes S$ to be quasi - ∗ - class A operator where T and S are both non-zero operators. Recall that $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ and so, by the uniqueness of positive square roots, $|T \otimes S|^r = |T|^r \otimes |S|^r$ for any positive rational number r. From the density of the rationales in the real, we obtain $|T \otimes S|' = |T|' \otimes |S|'$ for any positive real number p. If $T_1 \geq T_2$ and $S_1 \geq S_2$, then $T_1 \otimes S_1 \geq T_2 \otimes S_2$ (see, [17]).

Theorem 3.1.

Let S, T ∈ B(H) be non-zero normal operators. Then T ⊗ S is quasi - ∗ - class A normal operator if and only if one of the following holds:

a) S and T are quasi - ∗ - class A normal operators.

b) $S^2 = 0$ or $T^2 = 0$.

Proof.

Since T ⊗ S is quasi - ∗ - class A operator if and only if $(T \otimes S)^*(|(T \otimes S)|^2 - (|(T \otimes S)^*|))(T \otimes S) \geq 0 \Leftrightarrow T^*(|T^2| - |T^*|^2)T \otimes S^*|S^2|S + T^*|T^*|^2T \otimes S^*(|S^2| - |S^*|^2)S \geq 0$.Hence the sufficiency is clear. Conversely, assume that T ⊗ S is quasi - ∗ - class A operator. Then for every x, y ∈ H we have $(T^*(|T^2| - |T^*|^2)Tx,x)(S^*|S^2|Sy, y) + (T^*|T^*|^2Tx,x)(S^*(|S^2| - |S^*|^2)Sy, y) \geq 0$(3.1)

It suffices to prove that if (a) does not hold, then (b) holds. Suppose that T2 = 0 and $S^2 = 0$. To the contrary, assume that T is not a quasi - ∗ - class A operator, then there exists x0 ∈ H such that $(T^*(|T^2| - |T^*|^2)Tx0,x0) = \alpha < 0$ and $(T^*|T^*|^2Tx0, x0 = \beta > 0$. From (3.1) we have $\alpha + \beta(S^*|S^2|Sy, y) \geq \beta(S^*|S^*|^2Sy, y)$, for all y ∈ H. (3.2)

Thus S is quasi - ∗ - class A operator since α + β ≤ β. Using the H¨older-McCarthy inequality we have $(S^*|S^2|Sy,y) = ((S*^2S^2)^{1/2} Sy, Sy) \leq ||Sy||^{2(1/2)}(S^{*2}S^2Sy,Sy)^1/^2 = ||Sy||||S^3y||$ and$(S^*|S^*|^2Sy, y) = (SS^*Sy, Sy) = ((S^*S)y, S^*Sy) = ||S^*Sy||^2$. Thus, $(\alpha + \beta)||Sy||||S^3y|| \geq \beta||S^*Sy||^2$. (3.3)

Since S is a quasi - ∗ - class A operator, Lemma 2.1 imply that H = $ran(S^k) \oplus ker\ S^{*k}$.Then S1 is ∗ - class A, $S^k{}_3 = 0$ and σ(S) = σ(S1)∪{0}. Therefore (3.3) implies $(\alpha + \beta)||S1\eta||||S^3{}_1\eta|| \geq \beta||S^*{}_1S1\eta||^2$, for all $\eta \in$ ran S^k. Since S1 is ∗ - class A and ∗ - class A is normaloid. Thus taking supremum on both sides of the above inequality, we have $(\alpha + \beta)||S_1||^4 \geq \beta||S^*{}_1S_1||^2$.Therefore, $S_1 = 0$. Hence $S^{k+1} = 0$. This contradicts the assumption $S^2 = 0$. Hence T must be a quasi - * - class A operator. A similar argument shows that S is also quasi - * - class A normal operator. This completes the proof.

Corollary 3.2.

Let S^n, $T^n \in B(H)$ be non-zero normal power operators. Then $T^n \otimes S^n$ is quasi - ∗ - class A^n normal operator if and only if one of the following holds:

c) S^n and T^n are quasi - ∗ - class A normal operators.

d) Either $S^n = 0$ or $T^n = 0$.

Proof:

Follows from Theorem 3.1 and considering all non-zero natural number n greater than 2 for case b.

Conclusions

In the present work we have characterized Hilbert space operators which are Quasi - ∗ - class *A* normal operator. We have shown that if *S, T ∈ B(H)* are non-zero normal operators. Then *T ⊗ S* is quasi - ∗ - class A normal operator if and only if one of the following holds: S and T are quasi - ∗ - class A normal operators, and that either $S^2 = 0$ or $T^2 = 0$. These results are useful in classification oh Hilbrt space operators.

Conflicts of interest

The authors declare no conflict of interest.

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