Math 6378 - Symmetry

Review - PDE's

1st order Quasi Linear

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

Method of characteristic (MofC)

Solve $\quad \dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{c}$

solve in pairs to obtain

$$c_1 = \phi(x,y,u)$$

$$c_2 = \psi(x,y,u)$$

sol^n $\quad c_1 = f(c_2)$ or $c_2 = f(c_1)$

Ex 1 Solve $x u_x - y u_y = 1$ (Linear)

H of c $\quad \dfrac{dx}{x} = -\dfrac{dy}{y} = \dfrac{du}{1}$

1st pair $\quad \dfrac{dx}{x} = -\dfrac{dy}{y} \Rightarrow \ln x = -\ln y + \ln c_1$

$\Rightarrow c_1 = xy$

2nd pair $\quad \dfrac{dx}{x} = du \Rightarrow \ln x = u - c_2$

so $c_2 = u - \ln x$

sol^n $\quad c_2 = f(c_1) \Rightarrow u - \ln x = f(xy)$

or $\quad u = \ln x + f(xy)$

check $\quad u_x = \dfrac{1}{x} + f'(xy)\cdot y, \quad u_y = f'(xy)\cdot x$

LS $\quad x u_x - y u_y = x\left(\frac{1}{x} \cancel{+ y f'}\right) - y\left(\cancel{f' x}\right)$

$= 1$ ✓

Ex 2 $\quad u_x + u u_y = x \quad$ (quasi-Linear)

M of C $\quad \dfrac{dx}{1} = \dfrac{dy}{u} = \dfrac{du}{x}$

1st pair $\quad dx = \dfrac{du}{x} \quad$ or $\quad du = x\,dx$

$$u = x^2/2 + c_1$$

use 1st sol

2nd pair $\quad dx = \dfrac{dy}{u} \quad$ or $\quad dy = u\,dx$

$$= \left(\frac{x^2}{2} + c_1\right) dx$$

so $\quad y = \dfrac{x^3}{6} + c_1 x + c_2$

$$= \frac{x^3}{6} + \left(u - \frac{x^2}{2}\right) x + c_2$$

so $\quad y - xu + \frac{1}{3} x^3 = c_2$

so $\quad y - xu + \frac{1}{3} x^3 = f\left(u - x^2/2\right)$

or $\quad u - x^2/2 = g\left(y - xu + \frac{1}{3} x^3\right)$

Fully NonLinear

if $F(x, y, u, u_x, u_y) = 0$

if we let $p = u_x$, $q = u_y$ the method of characteristic is

$$
\left.\begin{aligned}
x_s &= F_p \\
y_s &= F_q \\
u_s &= pF_p + qF_q \\
p_s &= -F_x - pF_u \\
q_s &= -F_y - qF_u
\end{aligned}\right\}
$$

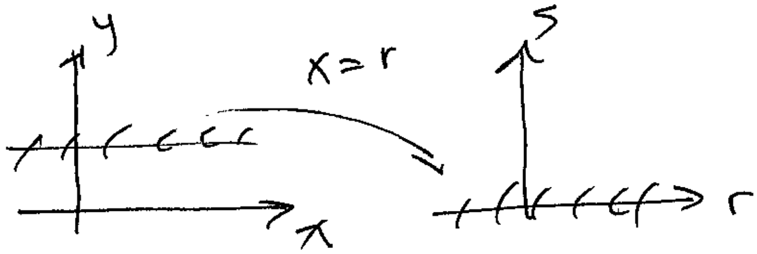
these are typically difficult to solve. Not so much in finding x, y, u, p & q but eliminating the parameter r & s

So we solve the PDE's subject to some BVP/IVP and we set up initial conditions at say $s=0$. The following example illustrates

Ex3 Soln

$$u_x u_y = 1 \quad \text{subject to} \quad u(x,1) = 2\sqrt{x}$$

So let $F = pq - 1$



so $x_s = F_p = q$

$y_s = F_q = p$

$u_s = pF_p + qF_q = 2pq = 2$

$p_s = 0$

$q_s = 0$

so B.C. $s=0$

$x = r$

$y = 1$

$u = 2\sqrt{r}$

Now from B.C. $u_x = \dfrac{1}{\sqrt{x}}$

& from PDE $u_y = \dfrac{1}{u_x} = \sqrt{x}$

so we solve these subject to

$s=0 \quad x=r,\ y=1,\ u=2\sqrt{r},\ p=\dfrac{1}{\sqrt{r}},\ q=\sqrt{r}$

① $p_s = 0 \Rightarrow p = a(r)$. $s=0 \ p = \dfrac{1}{\sqrt{r}} \Rightarrow p = \dfrac{1}{\sqrt{r}}$

② $q_s = 0 \Rightarrow q = b(r) \quad s=0 \ q = \sqrt{r} \Rightarrow q = \sqrt{r}$

③ $u_s = 2 \Rightarrow u = 2s + c(r) \quad s=0 \ u = 2\sqrt{r}$

so $u = 2s + 2\sqrt{r}$

④ $x_s = q = \sqrt{r}$ so $x = \sqrt{r}s + d(r)$

$s=0$ $x = r \Rightarrow d(r) = r$ so

$$x = \sqrt{r}s + r$$

⑤ $y_s = p = \frac{1}{\sqrt{r}}$ so $y = \frac{s}{\sqrt{r}} + e(r)$

$s=0$ $y=1 \Rightarrow e(r) = 1$ $y = \frac{s}{\sqrt{r}} + 1$

So now we ~~have~~ the sol^n parametrically

$$x = \sqrt{r}s + r, \quad y = \frac{s}{\sqrt{r}} + 1, \quad u = 2s + 2\sqrt{r}$$

$$= \sqrt{r}(s+\sqrt{r}) \qquad = \frac{s+\sqrt{r}}{\sqrt{r}}$$

so $\frac{x}{y} = \frac{\sqrt{r}\,\cancel{(s+\sqrt{r})}}{\frac{\cancel{s+\sqrt{r}}}{\sqrt{r}}} = r$ Now $s + \sqrt{r} = \sqrt{r}y$

$$u = 2s + 2\sqrt{r} = 2(\sqrt{r}y - \cancel{\sqrt{r}}) + \cancel{2\sqrt{r}}$$

$$= 2\sqrt{r}y = 2\sqrt{\frac{x}{y}}\cdot y = 2\sqrt{xy}$$

So our sol^n is $u = 2\sqrt{xy}$ Note $u(x,1) = 2\sqrt{x}$ ✓