Principle of Mathematical Induction

Let $P(n)$ be a statement that depends on $n \in \mathbb{N}$. If the following two conditions hold

1. $P(1)$ is true. (This is called the base case)
2. $P(k)$ is true implies $P(k+1)$ is true. (This is called the induction hypothesis and induction step)

then $P(n)$ is true for all $n \in \mathbb{N}$.

Example 1
Prove that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for $n \in \mathbb{N}$.

Solution
Since this is our first example we will be more tedious and show more steps than necessary.
Let $P(n)$ be the statement $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.
Base Case: $n = 1$
$P(1)$ is true because $1 = \frac{1(1+1)}{2}$.
Induction Hypothesis: Assume that $P(k)$ is true for some $k \in \mathbb{N}$. That is $1+2+3+\cdots+k = \frac{k(k+1)}{2}$.
Induction Step: Now, we will show that $P(k+1)$ is true.
Starting with the Induction Hypothesis, we add $(k+1)$ to both sides and algebraically manipulate the right side to get the desired result.

$$\begin{aligned} 1+2+3+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2k+2}{2} \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

Hence, $P(k+1)$ is true. Therefore, by mathematical induction, $1+2+3+\cdots+k = \frac{k(k+1)}{2}$ for all $n \in \mathbb{N}$.

Remark 1
This is a common type of problem where you are asked to prove the closed form formula. The only hard part is to get the right hand side to equal to the desired result. Instead of a summation, it could be a product or an recurrence relation. Sometimes, the problems requires you to first find a closed form formula then prove that it is correct.

Example 2
Prove that $2^{n-1} \leq n!$ for all $n \in \mathbb{N}$.

Solution
For this example, we will show less steps.
We will prove this inequality by induction on n.
Base Case: $n = 1$ is true because $1 \leq 1$.
Induction Step: Assume the result is true for $n = k$. We will prove $n = k+1$ is also true.
Starting with the induction hypothesis and multiply both sides of the inequality by 2, we have

$$\begin{aligned} 2^{(k+1)-1} &\leq 2 \cdot k! \\ &\leq (k+1)k! \\ &= (k+1)! \end{aligned}$$

Therefore by induction, $2^{n-1} \leq n!$ for all $n \in \mathbb{N}$.

Remark 2
This is another common type of problem where you are asked to prove an inequality. The only hard part to this example is to recognize the second inequality, $2 \leq k+1$.

Example 3
Show that $5^n - 1$ is divisible by 4 for every $n \in \mathbb{N}$.

Solution
We will prove this by induction on n.
Base Case: $n = 1$ is true because 4 divides $5^1 - 1 = 4$.
Induction Step: Assume the result is true for $n = k$. We will prove $n = k + 1$ is true.
Observe that

$$\begin{aligned} 5^{k+1} - 1 &= 5^{k+1} + 5^k - 5^k - 1 \\ &= 5^k(5-1) + (5^k - 1) \\ &= 4 \cdot 5^k + (5^k - 1) \end{aligned}$$

Since 4 is divisible by 4 and by induction hypothesis, $5^k - 1$ is divisible by 4. Thus, $5^{k+1} - 1$ is divisible by 4. Therefore, by induction, we have shown that $5^n - 1$ is divisible by 4 for all $n \in \mathbb{N}$.

Remark 3
Notice that the key point to this solution is to add 5^k and subtract 5^k. This is a common technique in mathematics.

Principle of Strong Mathematical Induction

There will be times when we are try to prove $P(k+1)$, we also need more than just $P(k)$ to be true. Therefore, we need the following:

Let $P(n)$ be a statement that depends on $n \in \mathbb{N}$. If the following two conditions hold

1. $P(1)$ is true. (This is called the base case)
2. $P(1), P(2), \ldots, P(k)$ are all true implies $P(k+1)$ is true for all $k \in \mathbb{N}$. (This is called the induction hypothesis and induction step)

then $P(n)$ is true for all $n \in \mathbb{N}$.

Example 3
Show that every $n \in \mathbb{N}_{>1}$ is either a prime or a product of primes.

Solution
We will use induction on the natural number $n > 1$.
Base Case: $n = 2$ is true because 2 is prime. (Notice that we are starting with 2 instead of 1)
Induction Step: Assume the result is true for $n = 2, 3, 4, \ldots, k$. We will show that $n = k + 1$ is also true.
If $k + 1$ is a prime then we are done. So assume that $k + 1$ is not prime. Then there exists $a, b > 1$ that are natural numbers such that $k + 1 = ab$. Since $1 < a, b < k + 1$, by induction hypothesis we can write a and b as a product of primes, $a = a_1 a_2 \cdots a_l$ and $b = b_1 b_2 \cdots b_m$ where each a_i and b_j is a prime. Hence, $k + 1 = a_1 \cdots a_i b_1 \cdots b_j$ is a product of primes. Therefore, by strong induction every $n \in \mathbb{N}_{>1}$ is either a prime or a product of primes.

Remark 4
Notice that an induction doesn't have to start at 1. In fact, it can even start from a negative number.

Practice Problems

1. Show that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.
2. Show that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{N}$.
3. Let $a_n = \frac{1}{2}a_{n-1} + \frac{1}{2}a_{n-2}$ for all $n \in \mathbb{N}_{\geq 3}$ with $a_1 = 3$ and $a_2 = \frac{3}{2}$. Show that $a_n = 2 + (-\frac{1}{2})^{n-1}$ for all $n \in \mathbb{N}$.
4. Show that $7^{2n} - 48n - 1$ is divisible by 2304 for every $n \in \mathbb{N}$.
5. Let $x > -1$ and $x \neq 0$. Show that $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.
6. Show that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n \in \mathbb{N}_{>1}$.
7. Let $a_1 = 1$ and for each $n \geq 1$ let $a_{n+1} = \sqrt{3 + 2a_n}$. Prove that for every $n \in \mathbb{N}$, we have $0 \leq a_n \leq a_{n+1} \leq 3$.
8. Prove that $2^n + 3^n$ is divisible by 5 for all odd $n \in \mathbb{N}$.
9. Prove that if $(x + \frac{1}{x}) \in \mathbb{N}$ then $(x^n + \frac{1}{x^n}) \in \mathbb{N}$ for all $n \in \mathbb{N}$.
10. Consider n lines on a plane such that no two are parallel and no three lines intersect at one point.
    (a) Find the closed form formula for the number of intersection points.
    (b) Find the closed form formula for the number of regions the plane has been split into.
11. Consider the following induction proof of "All horses are the same color". It is obvious this statement is false which means the proof is wrong. Find the error and explain why it is incorrect.

    Base Case: One horse. This case is trivial because there is only one horse. Thus, all horses are the same color.
    Induction Hypothesis: Assume that n horses always are the same color.
    Induction Step:
    First, exclude the last horse and look only at the first n horses. By induction hypothesis, all these n horses are the same color. Likewise, exclude the first horse and look only at the last n horses. These too, must also be of the same color. Therefore, the first horse in the group is of the same color as the horses in the middle, who in turn are of the same color as the last horse. Hence, the first horse, middle horses and last horse are all of the same color. Thus, all $n + 1$ horses are of the same color. Therefore, by induction, all horses are the same color.
12. Prove that $\underbrace{111\ldots1}_{3^n}$ is divisible by 3^n for all $n \in \mathbb{N}$.
13. Let $n \in \mathbb{N}$. Prove that if one square of a $2^n \times 2^n$ chessboard is removed, the remaining squares can be covered by L-shaped triminoes.
14. Suppose there are n identical cars on a circular track and among them there is enough gasoline for one car to make a complete loop around the track. Prove that there is one car that can make it around the track by collecting all of the gasoline from each car that it passes as it moves.
15. Let $c, p, q \in \mathbb{R}$ with $p \neq 0$. Let $a_0 = c$ and for $n \geq 1$ let $a_n = pa_{n-1} + q$. Find the closed form formula for a_n.
16. Suppose that $M = p_1 p_2 \cdots p_n$ where p_i are distinct primes. Prove that for all integers n, if $\gcd(a, M) = 1$ then
    $$a^{(p_1-1)(p_2-1)\cdots(p_n-1)} \equiv 1 \mod M$$
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