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bj

A Higgs Family: Part II

I. Introduction

In a previous note ("A Higgs Family: Part I"), we introduced a generalization of the standard "vanilla" Higgs model by embedding the usual degrees of freedom within an electroweak-doublet, family-nonet Higgs multiplet containing 36 self-conjugate degrees of freedom. Given a simple Higgs potential possessing an O(36) global symmetry, broken down to O(32) x O(4) by an additional "rogue" mass term, we constructed a phenomenology which appears to evade present-day experimental constraints. This phenomenology presumed that the 32 new degrees of freedom have masses in the range of 200-1000 GeV, kinematically accessible to the LHC. But it appears that discovering these particles at the LHC will not be simple. Much more detailed study of this issue is certainly called for. But progress may require imaginative ideas coupled with serious simulations.

However, the model only took into account the nonvanishing top quark mass, and neglected all other quark and lepton masses, as well as the concomitant mixings. Therefore, one can anticipate that this scheme will not be taken very seriously until these deficiencies are dealt with.

The purpose of this note is to take a first step in this direction. In this note, we include in the Higgs potential additional "induced" terms which must exist due to divergent closed-loop Feynman diagrams involving the quarks and leptons. These terms include complications associated with the known CKM mixing. What we find is on the whole encouraging: (1) In the limit of top-quark dominance, the three Goldstone modes, those to be eaten by the W and Z, remain massless. The spectrum of the 32 new states is modified, including some mixings of the charged degrees of freedom, but in a way that does not appear to grossly change the original phenomenology. (2) When the terms involving CKM mixing are included, the Goldstone modes do not remain massless unless the coefficient of the "rogue" mass term is set to zero. However, this is acceptable, because the new "induced" terms in the potential suffice to create the spontaneous symmetry breaking, while keeping the phenomenology of the 32 new degrees of freedom acceptable. (3) Once the "rogue" mass term is removed, inclusion of a nonvanishing b-quark mass does not create difficulties, even in the presence of the induced CKM mixing.

Nevertheless, these results do not shed much light on the fundamental mechanisms responsible for first and second generation masses and mixings. It seems probable that the Higgs sector needs further enlargement in order to make further progress. Consideration of this issue is beyond the scope of this note, but will hopefully be addressed in the next installment of this series.

II. The Higgs Sector and its Coupling to Quarks in the Standard Model

To establish notation and introduce the family-mixing description, we begin by reviewing the vanilla Higgs model in our language. We write the Higgs field in linear sigma-model language:

$$\Phi \equiv \frac{1}{\sqrt{2}}\left(H + i\vec{\tau}\cdot\vec{w}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} H + iw_3 & iw_1 + w_2 \\ iw_1 - w_2 & H - iw_3 \end{pmatrix} \qquad \bar{\Phi} = \frac{1}{\sqrt{2}}\left(H - i\vec{\tau}\cdot\vec{w}\right)$$

We have taken the gaugeless limit, so that the three w – fields are massless Goldstone modes. The more realistic version is beyond the scope of this note.

The Higgs potential is

$$V = -\frac{m^2}{2}\,\mathrm{Tr}\,\bar{\Phi}\Phi + \frac{\lambda}{4}\left(\mathrm{Tr}\,\bar{\Phi}\Phi\right)^2$$
$$= -\frac{m^2}{2}\left(H^2 + |\vec{w}|^2\right) + \frac{\lambda}{4}\left(H^2 + |\vec{w}|^2\right)^2$$

Of course, the trace goes only over electroweak Pauli matrices. As yet, there are no family indices in sight.

Of special interest is the Yukawa term which couples this Higgs sector to the quarks and leptons. We write

$$\mathcal{L}_{Yukawa} = \frac{\sqrt{2}}{v}\,\bar{Q}_L\,\Phi\, M Q_R + h.c.$$

$$Q_{L,R} \equiv \begin{pmatrix} U \\ D \end{pmatrix}_{L,R} = \begin{pmatrix} u \\ c \\ t \\ d \\ s \\ b \end{pmatrix}$$

This time the quarks are family triplets, and, as detailed in what follows, the mass term is built from a pair of 3 x 3 matrices in family space, each of which in general is neither real nor hermitian.

The Yukawa term must be invariant under left-handed chiral electroweak transformations, and under an abelian right-handed transformation associated with the electroweak-hypercharge degree of freedom:

$$Q_L \to e^{i\vec{\tau}\cdot\vec{\varphi}_L} Q_L \qquad Q_R \to e^{i\tau_3 \varphi_R} Q_R \qquad \Phi \to e^{i\vec{\tau}\cdot\vec{\varphi}_L} \Phi\, e^{-i\tau_3 \varphi_R}$$

Therefore the mass term is electroweak-diagonal:

$$M = \begin{pmatrix} M_{up} & 0 \\ 0 & M_{down} \end{pmatrix}$$

However, there are no restrictions on the structure of the Yukawa coupling coming from the U(3) x U(3) chiral family symmetry transformations. However, these transformations can be used to diagonalize the mass matrix. We write

$$\begin{pmatrix} U \\ D \end{pmatrix}_{L,R} = \begin{pmatrix} V(up) & 0 \\ 0 & V(down) \end{pmatrix}_{L,R} \begin{pmatrix} U' \\ D' \end{pmatrix}_{L,R} \qquad Q_{L,R} = V_{L,R}\, Q'_{L,R}$$

$$M = \begin{pmatrix} V_L(up) & 0 \\ 0 & V_L(down) \end{pmatrix} \begin{pmatrix} M'_{up} & 0 \\ 0 & M'_{down} \end{pmatrix} \begin{pmatrix} V_R^{-1}(up) & 0 \\ 0 & V_R^{-1}(down) \end{pmatrix} \qquad M = V_L M' V_R^{-1}$$

M'_{up} and M'_{down} are then real and diagonal matrices. The Yukawa coupling term then becomes

$$\mathcal{L}_{Yukawa} = \frac{1}{v} \bar{Q}'_L V_L^{-1} \Phi \left[V_L M' V_R^{-1} \right] V_R Q'_R + h.c. = \frac{1}{v} \bar{Q}'_L \Phi' M' Q'_R + h.c.$$

We have diagonalized the mass matrix at the expense of complicating the definition of Φ. It is therefore instructive to write it out in more detail:

$$\Phi' = V_L^{-1} \Phi V_L = \begin{pmatrix} V_L^{-1}(up) & 0 \\ 0 & V_L^{-1}(down) \end{pmatrix} \begin{pmatrix} H + i w_3 & i w_1 + w_2 \\ i w_1 - w_2 & H - i w_3 \end{pmatrix} \begin{pmatrix} V_L(up) & 0 \\ 0 & V_L(down) \end{pmatrix}$$

$$= \begin{pmatrix} (H + i w_3) & V_L^{-1}(up) V_L(down) (i w_1 + w_2) \\ V_L^{-1}(down) V_L(up) (i w_1 - w_2) & (H - i w_3) \end{pmatrix}$$

We here recognize the emergence of the CKM matrix:

$$V_{CKM} \equiv V_L^{-1}(up)\, V_L(down)$$

The vanilla Higgs model is notable because the complications of mixing do not affect the Higgs and w degrees of freedom. As is well recognized, this feature does not easily survive generalizations, including this one.

III. Review of the Higgs Family

In our previous note ("A Higgs Family: Part I"), we introduced a chiral family nonet of electroweak doublet Higgs fields. As mentioned already in the introduction, we found a very simple phenomenology for the extra 32 degrees of freedom, which appears to be consistent with experiment, provided we did not allow for first- and second-generation masses and mixings, or even the bottom-quark mass. Here we review that description in its most simple form:

The vanilla Higgs field is replaced with the nonet:

$$\Phi = \frac{1}{\sqrt{2}}\left(H + i\vec{\tau}\cdot\vec{\omega}\right) \quad \Rightarrow \quad \Phi_i^A \equiv \frac{1}{\sqrt{2}}\left(H_i^A + i\vec{\tau}\cdot\vec{\omega}_i^A\right)$$

The Yukawa term is now written

$$\mathcal{L}_{Yukawa} = \frac{\sqrt{2}}{V}\left(\bar{Q}_L\right)^i \Phi_i^B M_B^A \left(Q_R\right)_A$$

The Higgs potential is taken to be

$$V = \frac{m^2}{2}\mathrm{Tr}\,\bar{\Phi}\Phi + \frac{\lambda_0}{4}\left(\mathrm{Tr}\,\bar{\Phi}\Phi\right)^2 - \frac{m_3^2}{2}\mathrm{Tr}\,\bar{\Phi}_3^3\Phi_3^3$$

In the absence of the "rogue" mass term (the one with coefficient m_3^2), this potential has an O(36) global symmetry. The rogue term breaks this to O(32) x O(4), with the O(4) broken spontaneously, leaving in its wake the vanilla Higgs boson h_3^3 and the three Goldstone bosons $\vec{w}_3^3$ which get eaten by the W and Z gauge bosons. The remaining 32 degrees of freedom end up having a common mass, presumably in the 200-500 GeV range.

To see this, assume

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} H_3^3 &\equiv v + h_3^3 \\ H_i^A &\equiv h_i^A \text{ otherwise} \end{aligned}$$

Expanding out the potential and keeping the terms necessary to determine the masses of the dynamical degrees of freedom, we find

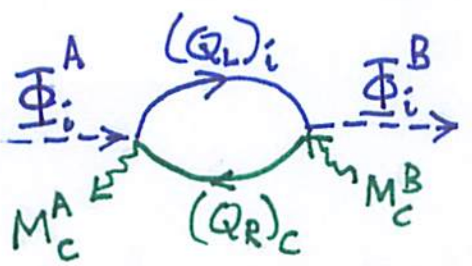
$$\begin{aligned} V = & \left[(m^2 - m_3^2)\frac{v^2}{2} + \frac{\lambda_0 v^4}{4} \right] + (h_3^3 v)^2 \left[(m^2 - m_3^2) + \lambda_0 v^2 \right] \\ & + \lambda_0 v^2 (h_3^3)^2 + \frac{1}{2} \operatorname{tr}(h^2 + |\vec{\omega}|^2)\left[m^2 + \lambda_0 v^2 \right] - \frac{m_3^2}{2}\left[(h_3^3)^2 + |\vec{\omega}_3^3|^2 \right] + \cdots \end{aligned}$$

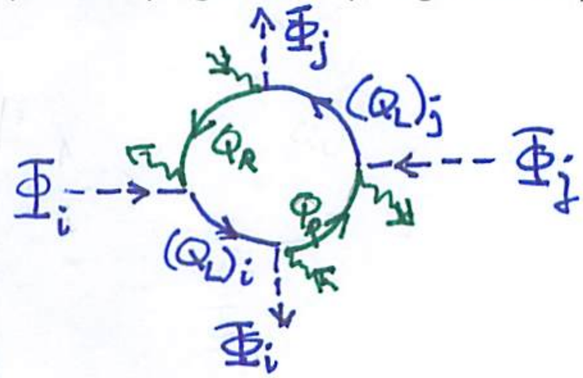
From this, we easily see that the 32 new degrees of freedom all have mass m_3, the $\vec{w}_3^3$'s are massless, and that the mass of the vanilla Higgs is

$$m_h^2 = 2(m_3^2 - m^2) = 2\lambda_0 v^2$$

IV. Loop Corrections

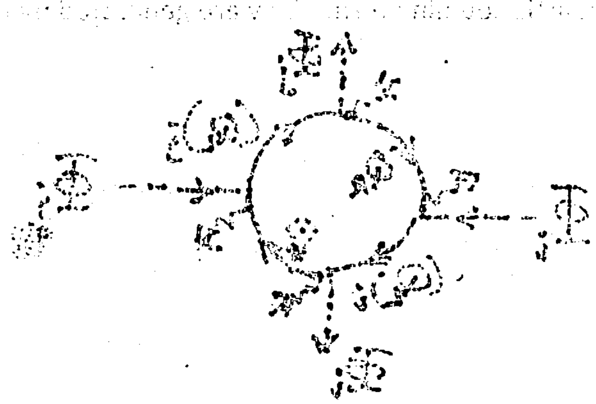
We now turn to the main purpose of this note---to include in the Higgs potential those symmetry breaking terms which we know for certain to exist, namely those generated by loop corrections involving the quarks and, eventually, the leptons. There are two such terms to consider---one quadratic mass term and one quartic coupling term. They are generated by the following Feynman diagrams:

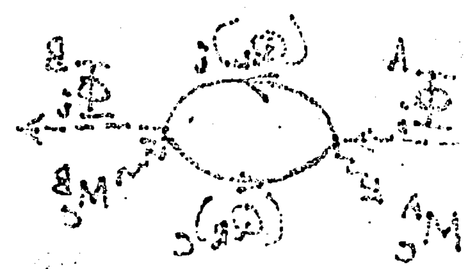




$$\mathcal{M} \sim \Phi_i^A \left(M_A^C M_C^{\dagger B} \right) \bar{\Phi}_B^i \int \frac{d^4k}{(2\pi)^4 k^2} \qquad \mathcal{M} \sim \Phi_i^A (MM^\dagger)_A^B \bar{\Phi}_B^j \Phi_j^C (MM^\dagger)_C^D \bar{\Phi}_D^i \int \frac{d^4k}{(2\pi)^4 k^4}$$

$$\sim \operatorname{Tr} \bar{\Phi}\Phi M M^\dagger \int \frac{d^4k}{(2\pi)^4 k^2} \qquad \sim \operatorname{Tr} \bar{\Phi}\Phi(MM^\dagger)\bar{\Phi}\Phi(MM^\dagger) \int \frac{d^4k}{(2\pi)^4 k^4}$$





The magnitude of the quadratic term is very uncertain because the coefficient is a quadratically divergent integral. It will be set by hand via purely phenomenological considerations. The quartic term, on the other hand, has as its coefficient a logarithmically divergent integral. Furthermore the top quark contribution is dominant, allowing an upper bound on the effective coupling constant λ_f.

$$|\lambda_f| \lesssim \left(\frac{m_t}{v}\right)^4 \cdot \overset{\text{color}}{3} \cdot \overset{\text{spin}}{4} \cdot \int \frac{d^4k}{(2\pi)^4} \cdot \frac{1}{k^4} \lesssim \frac{3}{2\pi^2}\left(\frac{m_t}{v}\right)^4 \ln \frac{m_{GUT}}{m_t} \lesssim 1.1$$

In any case these two terms may now be added to the original Higgs potential:

$$V = \frac{m^2}{2} \mathrm{Tr}\, \bar{\Phi}\Phi + \lambda_0 \left(\mathrm{Tr}\, \bar{\Phi}\Phi\right)^2 - \frac{m_3^2}{2} \mathrm{Tr}\, \bar{\Phi}_3^3 \Phi_3^3$$
$$- m_f^2 \,\mathrm{Tr}\, \bar{\Phi}\Phi\rho - \lambda_f \,\mathrm{Tr}\, \bar{\Phi}\Phi\rho\bar{\Phi}\Phi\rho$$

The matrix ρ is defined as

$$\rho = \frac{1}{m_t^2}\left(MM^\dagger\right) = \frac{1}{m_t^2}\begin{pmatrix} V_L^{-1}(up)\, M'(up)^2\, V_L(up) & 0 \\ 0 & V_L^{-1}(down)\, M'(down)^2\, V_L(down) \end{pmatrix}$$

In the limit of no mixing and top-quark dominance, ρ reduces to a projection operator, which we denote as P.

We now re-analyze the structure of the spontaneous symmetry breaking in the presence of the new terms. We first consider the limit of no mixing and top-quark dominance, so that $\rho = P$ is indeed a simple projection operator. Even in this limit, there occurs an interesting symmetry-breaking pattern. We begin by writing out the modified potential, as usual only to quadratic order:

$$V = \frac{m^2}{2}\left[\mathrm{tr}\left(H^2 + |\vec{\omega}|^2\right)\right] + \frac{\lambda_0}{4}\left[\mathrm{tr}\left(H^2 + |\vec{\omega}|^2\right)\right]^2 - \frac{m_3^2}{2}\left[\left(H_3^3\right)^2 + |\vec{\omega}_3^3|^2\right]$$
$$- \frac{m_f^2}{2}\left[\left(H^2 + |\vec{\omega}|^2\right)_3^3\right] - \frac{\lambda_f}{4}\mathrm{tr}\left\{\left[H^2 + |\vec{\omega}|^2 + i[H,\omega_3] + i[\omega_1,\omega_2]\right]P\right\}^2$$

$$\therefore V = \left[\frac{1}{2}\left(m^2 - m_3^2 - m_f^2\right)v^2 + \frac{1}{4}\left(\lambda_0 - \lambda_f\right)v^4\right] + \left(h_3^3 v\right)\left[\left(m^2 - m_3^2 - m_f^2\right) + \left(\lambda_0 - \lambda_f\right)v^2\right]$$
$$+ \frac{m^2}{2}\mathrm{tr}\left(h^2 + |\vec{\omega}|^2\right) - \frac{m_f^2}{2}\left(h^2 + |\vec{\omega}|^2\right)_3^3 - \frac{m_3^2}{2}\left[\left(h_3^3\right)^2 + |\vec{\omega}_3^3|^2\right] + \lambda_0 v^2\left[\left(h_3^3\right)^2 + \frac{1}{2}\mathrm{tr}\left(h^2 + |\vec{\omega}|^2\right)\right]$$
$$- \frac{\lambda_f v^2}{2}\left[\left(h^2 + |\vec{\omega}|^2\right)_3^3 + 2\left(h_3^3\right)^2 + \sum_{i=1}^{2}\left\{|h_3^i|^2 + |\vec{\omega}_3^i|^2 + i\left[(\omega_1)_i^3(\omega_2)_3^i - (\omega_2)_i^3(\omega_1)_3^i\right]\right\}\right] + \cdots$$

We can now proceed as usual to determine the spectrum of the 36 degrees of freedom:

The value of the vev is

$$(\lambda_0-\lambda_f)v^2 = (m_3^2 + m_f^2 - m^2)$$

The piece of the potential which yields a common mass for the 16 subgroup states is

$$V = \frac{1}{2}(m^2+\lambda_0 v^2)\left[(h_1^1)^2+(h_2^2)^2+2|h_1^2|^2+|\vec{\omega}_1^1|^2+|\vec{\omega}_2^2|^2+2|\vec{\omega}_1^2|^2\right]+\cdots$$

$$(mass)^2 = m^2+\lambda_0 v^2 = m_3^2+m_f^2+\lambda_f v^2$$

The corresponding terms for the 8 neutral coset states are

$$V = \left[\left(m^2-\frac{m_f^2}{2}\right)+(\lambda_0-\lambda_f)v^2\right]\left[|h_3^1|^2+|h_3^2|^2+|(\omega_3)_3^1|^2+|(\omega_3)_3^2|^2\right]+\cdots$$

$$(mass)^2 = m_3^2+\frac{m_f^2}{2}$$

The remaining 8 charged coset states mix. The mass eigenstates are split. The mass eigenfunctions can be written as follows (i = 1,2) :

$$(C_1)_i = \frac{1}{2}\left[(\omega_1-i\omega_2)_3^i+(\omega_1+i\omega_2)_i^3\right] \qquad (C_3)_i = \frac{1}{2}\left[(\omega_1+i\omega_2)_3^i+(\omega_1-i\omega_2)_i^3\right]$$

$$(C_2)_i = \frac{1}{2}\left[(\omega_1-i\omega_2)_3^i-(\omega_1+i\omega_2)_i^3\right] \qquad (C_4)_i = \frac{1}{2}\left[(\omega_1+i\omega_2)_3^i-(\omega_1-i\omega_2)_i^3\right]$$

Note that

$$\frac{1}{2}\left[C_1^2+C_2^2+C_3^2+C_4^2\right]_i = |(\omega_1)_3^i|^2+|(\omega_2)_3^i|^2$$

The corresponding piece of the Higgs potential is

$$V = \left[\left(m^2-\frac{m_f^2}{2}\right)+(\lambda_0-\lambda_f)v^2\right]\left[\sum_{i=1}^{2}\left(|(\omega_1)_3^i|^2+|(\omega_2)_3^i|^2\right)\right]-\frac{i\lambda_f v^2}{2}\sum_{i=1}^{2}\left[(\omega_1)_i^3(\omega_2)_3^i-(\omega_2)_i^3(\omega_1)_3^i\right]$$

$$= \sum_{i=1}^{2}\left\{\frac{1}{2}\left(m_3^2+\frac{m_f^2}{2}\right)\left[C_1^2+C_2^2+C_3^2+C_4^2\right]_i+\frac{1}{2}\lambda_f v^2\left[C_1^2+C_2^2-C_3^2-C_4^2\right]\right.$$

$$(mass)^2 = \left(m_3^2+\frac{m_f^2}{2}\right)\pm\lambda_f v^2$$

The three $\vec{w}_3^3$ states remain Goldstone:

$$V = \frac{1}{2}\left[(m^2-m_f^2-m_3^2)+(\lambda_0-\lambda_f)v^2\right]|\vec{\omega}_3^3|^2+\cdots = 0+\cdots$$

$$(mass)^2 = 0$$

Finally, the mass of the vanilla Higgs state h_3^3 is

$$V = \frac{1}{2}\left[(m^2-m_f^2-m_3^2)+3(\lambda_0-\lambda_f)v^2\right]|h_3^3|^2+\cdots$$

$$(mass)^2 = 2(\lambda_0-\lambda_f)v^2 = 2(m_3^2+m_f^2-m^2)$$

Before continuing, it is appropriate to add two quartic terms to the potential which were already considered in the previous note. We write

$$V = \frac{m^2}{2} \mathrm{Tr}\, \bar{\Phi}\Phi + \frac{\lambda_0}{4}\left(\mathrm{Tr}\, \bar{\Phi}\Phi\right)^2 - \frac{m_3^2}{2} \mathrm{Tr}\, \bar{\Phi}_3^3 \Phi_3^3$$

$$- m_f^2\, \mathrm{Tr}\, \bar{\Phi}\Phi\wp - \lambda_f \mathrm{Tr}\, \bar{\Phi}\Phi\wp\, \bar{\Phi}\Phi\wp$$

$$- \frac{\lambda_1}{4} \mathrm{Tr}\, \bar{\Phi}\Phi\bar{\Phi}\Phi - \frac{\lambda_2}{4} \mathrm{Tr}\, \tau_\alpha \bar{\Phi}\Phi\tau^\alpha \bar{\Phi}\Phi$$

The addition of these terms creates no new complications, and we will not go through the usual ritual in detail, but only quote the results:

The expression for the vev now reads

$$\left(m_3^2 + m_f^2 - m^2\right) = \left[\lambda_0 - \lambda_f - \frac{1}{2}(\lambda_1 + 3\lambda_2)\right] v^2$$

The common mass of the 16 subgroup states S is

$$(mass)_S^2 = m_3^2 + m_f^2 + \left[\lambda_f + \frac{1}{2}(\lambda_1 + 3\lambda_2)\right] v^2$$

The mass of the 8 neutral coset states G is

$$(mass)_G^2 = m_3^2 + \frac{m_f^2}{2}$$

The masses of the 8 charged coset states C break into two foursomes:

$$(mass)_C^2 = m_3^2 + \frac{m_f^2}{2} + (\lambda_2 \pm \lambda_f) v^2$$

The mass of the 3 Goldstone modes $\vec{w}_3^3$ remains zero:

$$(mass)_w^2 = 0$$

The mass of the vanilla Higgs boson h_3^3 is

$$(mass)_h^2 = 2\left(m_3^2 + m_f^2 - m^2\right)$$

An important feature of this structure is that we may remove the "rogue" mass term from the potential by setting m_3 to zero. Nothing dramatic happens to the spectrum provided $\lambda_2 >> \lambda_f$. Thus the new quartic term proportional to λ_2 does have special significance.

We have not included terms proportional to $(m_b / m_t)^2$ or $(m_b / m_t)^4$, which come from the expansion of the matrix $\wp$. As long as CKM mixing is ignored, the corrections appear to be innocuous.

V. Mixing

At this point, we have produced a reasonable extension of the vanilla Higgs model, with one glaring exception. CKM mixing has not yet been addressed, and it is not clear how the scheme responds to its introduction. This is of course a crucial issue, because if that mixing induces, e.g. family-symmetry-violating couplings to the Z (what is commonly called flavor-changing-neutral-currents, or FCNC for short), there are very severe phenomenological constraints to address.

We will approach this issue a step at a time. We will begin by setting most of the quartic couplings to zero. But we will keep the quadratic couplings intact, just to see what happens. We will also set the bottom-quark mass to zero as well. Therefore

$$V = \frac{m^2}{2}\,\mathrm{Tr}\,\bar{\Phi}\Phi + \frac{\lambda_0}{4}\left(\mathrm{Tr}\,\bar{\Phi}\Phi\right)^2 - \frac{m_3^2}{2}\,\mathrm{Tr}\,\bar{\Phi}_3^3\Phi_3^3 - m_f^2\,\mathrm{Tr}\,\bar{\Phi}\Phi\mathcal{P}$$

$$\mathcal{P} = \begin{pmatrix} VPV^{-1} & 0 \\ 0 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The mixing matrix V only creates a small amount of mixing of the second and third generation fields:

$$V \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \qquad VPV^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

We now examine the effect of the mixing on the phenomenology. We assume that the Higgs vev is rotated in a similar way, but that the $\vec{w}$ fields do not acquire a vev:

$$\langle H\rangle = v\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\phi & \cos\phi\sin\phi \\ 0 & \cos\phi\sin\phi & \cos^2\phi \end{pmatrix} \equiv \tilde{v} \qquad H = \tilde{v} + h$$

Since θ is small compared to unity, we anticipate that the same will be true of ϕ.

The dependence of the potential on the vacuum parameters is

$$V = \frac{m^2 v^2}{2} + \frac{\lambda_0 v^4}{4} - \frac{m_3^2}{2}v^2\cos^2\phi - \frac{m_f^2}{2}v^2\cos^2(\phi-\theta)$$

The minimum conditions are

$$\delta\phi: \qquad m_3^2 \cos\phi \sin\phi + m_f^2 \cos(\phi-\theta)\sin(\phi-\theta) = 0$$

$$\delta v: \qquad \lambda_0 v^2 = m_3^2 \cos^2\phi + m_f^2 \cos^2(\phi-\theta) - m^2$$

Given that all these mixing angles are small compared to unity, the conditions on the angles simplify considerably:

$$\phi \approx \frac{m_f^2}{(m_3^2+m_f^2)}\theta \qquad\qquad (\phi-\theta) \cong \frac{m_3^2}{(m_3^2+m_f^2)}\theta$$

The value of the vev is only slightly perturbed.

$$\lambda_0 v^2 \cong (m_3^2 + m_f^2 - m^2) - \frac{m_3^2 m_f^2 \theta^2}{(m_3^2 + m_f^2)^2}$$

We now can expand the potential to quadratic order in the dynamical fields and see what happens to the Goldstone modes:

$$\begin{aligned} V = & \left[\frac{m^2 v^2}{2} + \frac{\lambda_0 v^4}{4} - \frac{m_3^2 v^2 \cos^2\phi}{2} - \frac{m_f^2 v^2}{2}\cos^2(\phi-\theta)\right] \\ & + \left[(m^2+\lambda_0 v^2)\,\mathrm{tr}\,\tilde{v}h - m_3^2\, \tilde{v}_3^3 h_3^3 - m_f^2\,\mathrm{tr}\,\tilde{v}h\rho\right] \\ & + \frac{1}{2}\left\{(m^2+\lambda_0 v^2)\,\mathrm{tr}(h^2+|\vec{w}|^2) - m_3^2\left[(h_3^3)^2+|\vec{w}_3^3|^2\right] + \lambda_0(\mathrm{tr}\,h\tilde{v})^2 - \frac{m_f^2}{2}\mathrm{tr}(h^2+|\vec{w}|^2)\rho\right\} + \cdots \end{aligned}$$

There is no mixing between the h degrees of freedom and the $\vec{w}$ degrees of freedom. We hereafter set the h degrees of freedom aside. We also see that the w_1, w_2, and w_3 degrees of freedom all behave the same way. So there are either three Goldstone modes or none. Hereafter we omit the vector that should decorate the tops of all the $\vec{w}$ degrees of freedom appearing in the equations of this section.

The w degrees of freedom involving the first generation, i.e. those containing at least one 1-index, are all massive and clearly decouple from the Goldstone candidates. And of the remaining four degrees of freedom, one of them also decouples, namely the imaginary part of w_2^3. Therefore only three remain. We rename them as follows:

$$\begin{aligned} \omega_2 &\equiv w_2^2 \\ \omega &\equiv \frac{1}{\sqrt{2}}(w_2^3+w_3^2) \\ \omega_3 &\equiv w_3^3 \end{aligned} \qquad\qquad \left[\ \hat{\omega} = \frac{1}{\sqrt{2}}(w_2^3-w_3^2) \text{ decouples}\ \right]$$

The piece of the potential which involves these three degrees of freedom reduces to the following;

$$V = \frac{1}{2}(m^2+\lambda_0 v^2)\left[(\omega_2)^2+(\omega_3)^2+\omega^2\right] - \frac{1}{2}m_3^2(\omega_3^2)$$
$$-\frac{m_f^2}{2}\left[(\omega_2)^2\sin^2\theta + \frac{\omega^2}{2}\sin^2\theta + (\omega_3)^2\cos^2\theta + \frac{\omega^2}{2}\cos^2\theta + (\omega_2+\omega_3)\omega\sqrt{2}\sin\theta\cos\theta\right]$$

We may test for Goldstone behavior without diagonalization by constructing the 3 x 3 mass matrix of these degrees of freedom. It is

$$V = \begin{pmatrix}\omega_2 & \omega & \omega_3\end{pmatrix}\begin{pmatrix}(m^2+\lambda_0 v^2 - m_f^2\sin^2\theta) & -\frac{m_f^2}{\sqrt{2}}\cos\theta\sin\theta & 0 \\ -\frac{m_f^2}{\sqrt{2}}\cos\theta\sin\theta & (m^2+\lambda_0 v^2-\frac{m_f^2}{2}) & -\frac{m_f^2}{2}\cos\theta\sin\theta \\ 0 & -\frac{m_f^2}{\sqrt{2}}\cos\theta\sin\theta & (m^2+\lambda_0 v^2 - m_3^2 - m_f^2\cos\theta)\end{pmatrix}\begin{pmatrix}\omega_2\\ \omega\\ \omega_3\end{pmatrix}$$

If Goldstone modes persist, the determinant of this mass matrix should vanish:

$$\left\|\begin{matrix}(m_3^2\cos^2\phi + m_f^2\cos^2(\phi-\theta) - m_f^2\sin^2\theta) & -\frac{m_f}{\sqrt{2}}\cos\theta\sin\theta & 0 \\ -\frac{m_f}{\sqrt{2}}\cos\theta\sin\theta & (m_3^2\cos^2\phi + m_f^2\cos^2(\phi-\theta) - \frac{m_f^2}{2}) & -\frac{m_f}{\sqrt{2}}\cos\theta\sin\theta \\ 0 & -\frac{m_f}{\sqrt{2}}\cos\theta\sin\theta & (-m_3^2\sin^2\phi + m_f^2\cos^2(\phi-\theta) - m_f^2\cos^2\theta)\end{matrix}\right\| \overset{?}{=} 0$$

Since the lower right diagonal element is of order θ^2, and all off-diagonal elements are of order θ, it is easily seen that the condition that the determinant vanish through order θ^2 is that the 2 x 2 subdeterminant in the lower right-hand corner should vanish at order θ^2. However, direct calculation shows that this is not the case:

$$\left\|\begin{matrix}(m_3^2+\frac{1}{2}m_f^2) & -\frac{m_f^2\theta}{\sqrt{2}} \\ -\frac{m_f^2\theta}{\sqrt{2}} & (-m_3^2\phi^2 - m_f^2(\phi-\theta)^2 + m_f^2\theta^2)\end{matrix}\right\| \cong \frac{m_3^2 m_f^4}{2(m_3^2+m_f^2)} \neq 0$$

Therefore we hereafter set m_3 to zero. In the limit of vanishing b-quark mass, any mixing in the up-quark sector can be compensated by a family rotation of Φ, leading to a satisfactory phenomenology. But, in going further, there are two questions which need to be answered. The first is whether introduction of a nonvanishing b-quark mass, but without mixing, still leaves the Goldstone modes $\vec{w}_3^{\,3}$ massless. If this question is answered in the affirmative, the second question is whether this conclusion survives the inclusion of CKM mixing.

We shall address these together. We write for the potential

$$V = \frac{m^2}{2}\mathrm{Tr}\,\bar{\Phi}\Phi + \frac{\lambda_0}{4}\left(\mathrm{Tr}\,\bar{\Phi}\Phi\right)^2 - m_f^2\,\mathrm{Tr}\,\bar{\Phi}\Phi\mathcal{P} - \lambda_f\,\mathrm{Tr}\,\bar{\Phi}\Phi\mathcal{P}\bar{\Phi}\Phi\mathcal{P}$$

$$= \frac{1}{2}m^2\,tr\left(H^2+|\vec{\omega}|^2\right) + \frac{\lambda_0}{4}\left[tr\left(H^2+|\vec{\omega}|^2\right)\right]^2 - \frac{1}{2}m_f^2\,tr\left(H^2+|\vec{\omega}|^2\right)\left(P+\epsilon^2P'\right)$$

$$-\frac{\lambda_f}{4}tr\left\{\left[H^2+|\vec{\omega}|^2+i[H,\omega_3]+i[\omega_1,\omega_2]\right]P\right\}^2$$

$$-\frac{\lambda_f}{4}\epsilon^4 tr\left\{\left[H^2+|\omega|^2-i[H,\omega_3]-i[\omega_1,\omega_2]\right]P'\right\}^2+\frac{\lambda_f\epsilon^2}{2}tr\left\{[H,(\omega_1-i\omega_2)]P'[H,(\omega_1+i\omega_2)]P\right\}$$

The quantities $\mathcal{P}$, P, P', and ϵ are defined as follows

$$\mathcal{P}=\begin{pmatrix} P & 0 \\ 0 & \epsilon^2 P' \end{pmatrix} \quad P=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad P'=UPU^{-1} \quad U=V_LV_R^{-1}=U_{CKM} \quad \epsilon=\left(\frac{m_b}{m_t}\right)$$

Note that we have rotated the fields Φ so that the mixing only appears when the bottom-quark mass is set to a nonzero value. Note also that we do not attempt to account for second-generation masses, but only test for the consistency of nonvanishing mixing in the extended Higgs sector.

We now can, as usual, expand the potential to second order in the dynamical fields. We again assume that the vev, now a matrix, occurs only within the H degrees of freedom. We find

$$V=\frac{m^2}{2}tr\,v^2+\frac{\lambda_0}{4}\left(tr\,v^2\right)^2-\frac{m_f^2}{2}tr\left[v^2\left(P+\epsilon^2P'\right)\right]-\frac{\lambda_f}{4}tr\left[v^2Pv^2P+\epsilon^4v^2P'v^2P'\right]$$

$$+\left[m^2tr(vh)+\lambda_0\left(tr\,v^2\right)tr(vh)-\frac{m_f^2}{2}tr\{v,h\}\left(P+\epsilon^2P'\right)-\frac{\lambda_f}{2}tr\,v^2\left(P\{v,h\}P+\epsilon^4P'\{v,h\}P'\right)\right]$$

$$+\frac{m^2}{2}tr\left(h^2+|\vec{\omega}|^2\right)+\frac{\lambda_0}{2}\left(tr\,v^2\right)tr\left(h^2+|\vec{\omega}|^2\right)+\lambda_0\left(tr\,hv\right)^2-\frac{m_f^2}{2}tr\left[\left(h^2+|\vec{\omega}|^2\right)\left(P+\epsilon^2P'\right)\right]$$

$$-\frac{\lambda_f}{2}tr\,v^2\left[P|\vec{\omega}|^2P+\epsilon^4P'|\vec{\omega}|^2P'\right]-i\frac{\lambda_f}{2}\left(tr\,v^2\right)\left[P[\omega_1,\omega_2]P-\epsilon^4P'[\omega_1,\omega_2]P'\right]$$

$$-\frac{\lambda_f}{4}\left\{tr\left[\{v,h\}+i[v,\omega_3]\right]P\right\}^2-\frac{\lambda_f}{4}\epsilon^4\left\{tr\left[\{v,h\}-i[v,\omega_3]\right]P'\right\}^2+\frac{\lambda_f\epsilon^2}{2}tr\left\{[v,(\omega_1-i\omega_2)]P'[v,(\omega_1+i\omega_2)]P\right\}$$

We shall assume that the vev v is also proportional to a projection operator:

$$\langle H\rangle = vP'' = v\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\phi & \sin\phi\cos\phi \\ 0 & \sin\phi\cos\phi & \cos^2\phi \end{pmatrix} \qquad P' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & \sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

The potential, as function of the vev, is

$$\langle V\rangle = \frac{m^2v^2}{2} + \frac{\lambda_0 v^4}{2} - \frac{m_f^2 v^2}{2}\, tr\, P''(P+\epsilon^2 P') - \frac{\lambda_f v^4}{4}\, tr\, P''(PP''P + \epsilon^4 P'P''P')$$

$$= \frac{v^2}{2}\left[m^2 - m_f^2\left(\cos^2\phi + \epsilon^2\cos^2(\phi-\theta)\right)\right] + \frac{v^4}{4}\left[\lambda_0 - \lambda_f\left(\cos^4\phi + \epsilon^4\cos^4(\phi-\theta)\right)\right]$$

The minimum conditions then reduce to the expressions

$$\delta\phi: \qquad m_f^2\left[\cos\phi\sin\phi + \epsilon^2\cos(\phi-\theta)\sin(\phi-\theta)\right] + \lambda_f v^2\left[\cos^3\phi\sin\phi + \epsilon^4\cos^3(\phi-\theta)\sin(\phi-\theta)\right] = 0$$

$$\delta v: \qquad m^2 - m_f^2\left[\cos^2\phi + \epsilon^2\cos^2(\phi-\theta)\right] + v^2\left\{\lambda_0 - \lambda_f\left[\cos^4\phi + \epsilon^4\cos^4(\phi-\theta)\right]\right\} = 0$$

To good approximation, we find that the corrections to the vev v are very small, and that

$$\phi \approx \epsilon^2\theta\,\frac{m_f^2}{(m_f^2 + \lambda_f v^2)}$$

However, when it comes to the Goldstone modes, we need to avoid approximations. Assume that the Goldstone modes, in family space, are proportional to P'' :

$$\vec{\omega}_i^{\,j} = \vec{\omega}_G (P'')_i^j + (\vec{\omega}_\perp)_i^j$$

$$tr\ P''\vec{\omega}_\perp = 0 \qquad\qquad tr\ P''\vec{\omega} = \vec{\omega}_G$$

Upon examination of the form of the potential, we must pay attention to possible cross terms between $\vec{w}_G$ and $\vec{w}_\perp$. We find that the relevant terms involving the Goldstone bosons are as follows

$$V = \frac{|\vec{\omega}_G|^2}{2}\left[m^2 + \lambda_0 v^2 - m_f^2\, tr P''(P+\epsilon^2 P') - \lambda_f v^2 (PP''P + \epsilon^4 P'P''P')\right]$$

$$-\frac{\vec{\omega}_G}{2}\cdot tr\{P'', \vec{\omega}_\perp\}\left[m_f^2(P+\epsilon^2 P') + \lambda_f v^2 (PP''P + \epsilon^4 P'P''P')\right]$$

$$= \frac{|\vec{\omega}_G|^2}{2}\left[m^2 + \lambda_0 v^2 - m_f^2\left(\cos^2\phi + \epsilon^2\cos^2(\phi-\theta)\right) - \lambda_f v^2\left(\cos^4\phi + \epsilon^4\cos^4(\phi-\theta)\right)\right]$$

$$-\frac{\vec{\omega}_G}{2}\cdot(\vec{\omega}_2^3 + \vec{\omega}_3^2)\left\{m_f^2\left[\sin\phi\cos\phi + \epsilon^2\sin(\phi-\theta)\cos(\phi-\theta)\right] + \lambda_f v^2\left[\sin\phi\cos^3\phi + \epsilon^4\sin(\phi-\theta)\cos^3(\phi-\theta)\right]\right\} + \dots$$

$$= 0 + \dots !!$$

The last line of this expression requires some explanation. In that line, we have chosen a basis for which P'' is diagonal. It is in that basis that the traces are most easily evaluated.

The bottom line is that, even in the presence of mixing and of nonvanishing bottom-quark mass, the Goldstone modes survive.

In addition, the remaining family members will undergo small mass shifts and mixings. We do not document those changes here. What we have found is that the addition of the bottom quark mass and the subsequent mixings does not upset the general picture. And the magnitude of the mixing is very small, characterized by the small parameter

$$\phi \lesssim \epsilon^2 \theta \sim \left(\frac{m_b}{m_t}\right)^2 V_{cb} \sim 3 \times 10^{-5}$$

VI. Final Comments

In this note, we have included the terms in the Higgs potential which must be there via radiative corrections involving fermion loops. We have found that, provided the original "rogue" mass term is eliminated, and provided that all the quartic terms in the potential considered in the previous note are included, the general consistency of the scheme seems to be preserved.

However, the origin of the first-and second-generation masses, and of the CKM mixings, are not really directly addressed. It appears that these require more input. This problem is beyond the scope of this note. Hopefully, it will be addressed in a sequel.