8 Sept 2011
by

Gauss-Bonnet Flow: Where Does it Go?

I. Introduction

In the last section of my pdf note "The MacDowell-Mansouri Extension: Addendum", a coordinate-patch description of Schwarzschild-deSitter spacetime using the Painleve-Gullstrand (PG) metric is discussed. The PG flow velocity is naturally inward in the region dominated by the black hole source in the neighborhood of the origin of coordinates. At large radii, where dark energy dominates the spacetime curvature, the PG velocity field is outward, in accordance with the accelerated expansion of the universe. At the patch boundary, called the rift zone, the velocity is discontinuous; it changes sign. While this does not affect the behavior of classical observables, e.g. test particles, it might still have significance.

Two speculative conditions are sufficient for this to be the case. One is that the Gauss-Bonnet (GB) "topological charge", with its huge MM coefficient, is a relevant descriptive element. The other is that the gravitation theory is emergent, in the sense of condensed-matter analog gravity. In such a case, the PG metric description may represent a preferred set of coordinates, with the velocity field associated with some kind of vacuum condensate. The GB topological charge is odd in the PG velocity field, unlike the elements of the Riemann tensor. So if it somehow is a condensate property, and finds its way into observables, the rift zones may be spacetime regions which contain new physics.

We also identified "subduction zones", located just outside sources of nuclear matter density. Within these boundary regions, the MM description arguably breaks down. In my MM pdf's, I speculate regarding what might happen there. Again, the subduction zones might conceivably be candidate spacetime regions containing new physics. However, such speculations lie outside the material needed for this note.

II. A Simple Model

No matter what, all this is admittedly very speculative. And one of the weaknesses of the idea is that it is wedded to a very special kind of metric. So the question naturally arises as to whether the description can be generalized beyond PG metrics. This note is devoted to a baby step in that direction. The central question is a simple one: how are the equations governing the PG flow velocity related to the source properties? For example, if the source were the earth, surrounded by nothing but deSitter space (dark energy only), what would happen to the PG flow in the interior of the earth? Near the center we expect it to go to zero. But what are the equations governing the decrease in the flow as one goes inward? We will choose a system even simpler than Planet Earth. It is essentially a Christmas tree ornament: a hollow sphere of material of normal density. For definiteness, let it have a radius of 2 cm and a thickness of 1 mm, and assume that the mass distribution is a continuum (Loosening this criterion is interesting and will be mentioned in a later section.)

Such a system cannot be described in PG terms alone. A natural generalization is to appeal to the ADM Hamiltonian description of classical gravity. There the PG velocity field is essentially what is known as "shift". But in ADM it is accompanied by a variable N called "lapse". In addition, in the general case there appear additional degrees of freedom, associated with intrinsic space curvature (Lapse and shift have to do with extrinsic curvature.) But for our purposes here, they can be ignored. This means that the modified ADM/PG metric can be written as follows:

$$ds^2 = N^2(r)\,dt^2 - (dr - v(r)\,dt)^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The lapse will be a function of radius r . But we will ignore time dependence. What we will find is that there needs to be external pressure applied to keep the source material comprising the sphere from collapsing.

But things simplify in a quasi-nonrelativistic limit, where only terms quadratic or less in the PG velocity (or shift) are retained. The necessary applied pressure turns out to be fourth order in v. So much of the story which follows, but not all, will look rather Newtonian.

Before going ahead with the equations, it is fun to put in some numbers. The mass of the sphere is

$$M \approx \left(\frac{6\times 10^{23}\,\text{GeV}}{\text{cm}^3}\right) \times 4\pi\,(2\ \text{cm.})^2 \times (1\ \text{mm.}) \approx 3\times 10^{24}\,\text{GeV}$$

The rift-surface radius surrounding this sphere, beyond which the "condensate" flows outward, occurs at the following radius

$$r^3 \sim \frac{GM}{H^2} \sim \frac{(3\times 10^{24}\,\text{GeV})\times(10^{28}\,\text{cm.})^2}{(1.6\times 10^{19}\,\text{GeV})^2} \sim (30{,}000\ \text{km.})^3$$

The PG velocity at the surface of the sphere is

$$V_0^2 \sim \frac{GM}{R} \sim \frac{(3 \times 10^{24}\, GeV)}{(1.6 \times 10^{19}\, GeV)^2 \times (2\ cm.)} \sim \left(3 \times 10^{-4} \frac{cm}{sec}\right)^2$$

Therefore the time the sphere takes to collapse under its own gravity (in the absence of external forces to hold the matter in place is

$$t \sim \frac{R}{V_0} \sim \frac{(2\ cm.)}{(3 \times 10^{-4}\, cm/sec.)} \sim 100\ min$$

III. Some Equations

With the above expression for the metric and assuming spherical symmetry, standard computations lead to the following form for the Riemann tensor.

$$R^{\alpha\beta}_{\mu\nu} = \begin{array}{c} tr \\ t\theta \\ t\varphi \\ \theta\varphi \\ r\varphi \\ r\theta \\ \\ \end{array} \begin{array}{c} \left(\begin{array}{cccccc} \left(\frac{u''}{2} - N''\right) & 0 & 0 & & & \\ 0 & \left(\frac{u'}{2r} - \frac{N'}{r}\right) & 0 & & 0 & \\ 0 & 0 & \left(\frac{u'}{2r} - \frac{N'}{r}\right) & & & \\ & & & \frac{u}{r^2} & 0 & 0 \\ & 0 & & 0 & \frac{u'}{2r} & 0 \\ & & & 0 & 0 & \frac{u'}{2r} \end{array} \right) \\ \begin{array}{cccccc} tr & t\theta & t\varphi & \theta\varphi & r\varphi & r\theta \end{array} \end{array}$$

where

$$u = V^2$$

The passage from this Riemann tensor to the Einstein tensor is straightforward:

Ricci tensor:

$$R_\mu^{\ \nu} = \begin{array}{c} t \\ r \\ \theta \\ \varphi \\ \\ \end{array} \begin{pmatrix} \left(\frac{u''}{2}+\frac{u'}{r}-N''-\frac{2N'}{r}\right) & 0 & 0 & 0 \\ 0 & \left(\frac{u''}{2}+\frac{u'}{r}-N''\right) & 0 & 0 \\ 0 & 0 & \left(\frac{u}{r^2}+\frac{u'}{r}-\frac{N'}{r}\right) & 0 \\ 0 & 0 & 0 & \left(\frac{u}{r^2}+\frac{u'}{r}-\frac{N'}{r}\right) \end{pmatrix}$$

$$\begin{array}{cccc} t & r & \theta & \varphi \end{array}$$

Ricci scalar;

$$R = \left(u'' + \frac{4u'}{r} + \frac{2u}{r^2} - 2N'' - \frac{4N'}{r}\right)$$

Einstein tensor:

$$G_\mu^{\ \nu} = \begin{array}{c} t \\ r \\ \theta \\ \varphi \end{array} \begin{pmatrix} \left(\frac{u'}{r}+\frac{u}{r^2}\right) & 0 & 0 & 0 \\ 0 & \left(\frac{u'}{r}+\frac{u}{r^2}-\frac{2N'}{r}\right) & 0 & 0 \\ 0 & 0 & \left(\frac{u''}{2}+\frac{u'}{r}-\frac{N'}{r}-N''\right) & 0 \\ 0 & 0 & 0 & \left(\frac{u''}{2}+\frac{u'}{r}-\frac{N'}{r}-N''\right) \end{pmatrix}$$

$$\begin{array}{cccc} t & r & \theta & \varphi \end{array}$$

Given the premise that the pressure is of order v^4, this yields three Einstein equations:

$$\frac{u'}{r} + \frac{u}{r^2} \cong 8\pi G \rho(r)$$

$$\frac{u'}{r} + \frac{u}{r^2} \cong \frac{2N'}{r}$$

$$\frac{u''}{2} + \frac{u'}{r} \cong N'' + \frac{N'}{r}$$

The third equation is obtainable by differentiation of the second equation. The second equation can be simply rewritten as

$$N' \cong 4\pi G r \rho(r)$$

Outside the spherical source of mass, the PG metric is valid, and the lapse $N = 1$. Therefore

$$N \cong 1 - 4\pi G \int_r^\infty dr\, r\, \rho(r)$$

The first Einstein equation determines the PG "condensate" velocity field:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(ru) \cong 8\pi G \rho(r)$$

$$u = V^2 = \frac{2GM(r)}{r}$$

where

$$M(r) = \int_0^r dr \left[4\pi r^2 \rho(r) \right]$$

For the record, the gravitational potential Φ is determined by Newton's second law:

$$\frac{\partial \Phi}{\partial r} = \frac{GM(r)}{r^2}$$

Since Φ tends to zero at the rift surface, we have

$$\Phi = -\int_r^\infty \frac{dr}{r^2} GM(r)$$

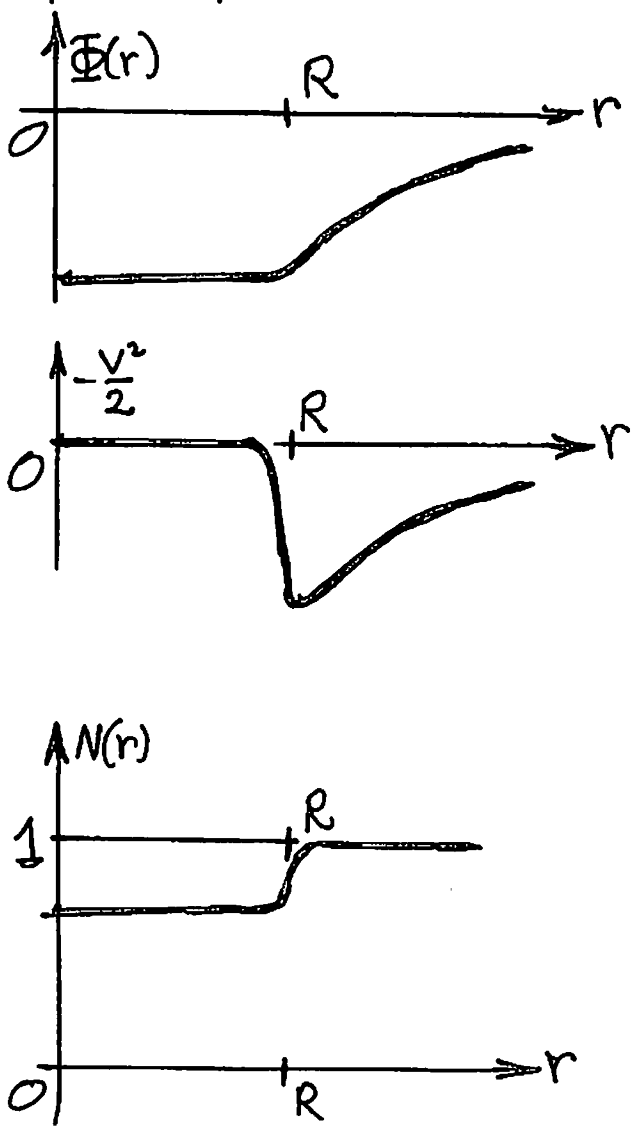
This allows an interesting expression for the lapse:

$$N \cong 1 - G\int_r^{\infty} \frac{dr}{r}\frac{\partial M}{\partial r} = 1 - \frac{GM(r)}{r}\Bigg|_r^{\infty} - G\int_r^{\infty}\frac{dr}{r^2}M(r)$$

$$= 1 + \frac{GM(r)}{r} + \Phi(r)$$

$$= 1 + \frac{v^2}{2} + \Phi(r)$$

These results are depicted in the plots below:



IV. Thermogravity

We see from the preceding equations that outside the spherical source the "elements of the condensate" fall in the same way as a massive test particle released from rest at infinity. However, within the sphere this is not so. Unlike the test particle, a condensate element is decelerated to rest by the time it reaches the interior of the sphere.

The equation of motion of the condensate element, at least for this highly symmetric geometry, is

$$\frac{d}{dr}\left(\frac{v^2}{2}\right) = v\frac{dv}{dr} = \frac{\partial v}{\partial t} \equiv -\frac{\partial V_{PG}}{\partial r} = \frac{\partial}{\partial r}\frac{GM(r)}{r}$$

On the other hand, the equation of motion for the test particle is

$$\frac{\partial v}{\partial t} = -\frac{\partial \Phi}{\partial r} = -\frac{GM(r)}{r^2}$$

The difference of the accelerations is related to the nontriviality of the lapse function:

$$\left(\frac{dv}{dt}\right)_{PG} - \left(\frac{dv}{dt}\right)_{\substack{test\\particle}} = \frac{\partial}{\partial r}\left(\frac{GM(r)}{r}\right) + \frac{GM(r)}{r^2} = \frac{G}{r}\frac{\partial M}{\partial r} = 4\pi G r \rho = \frac{\partial N}{\partial r}$$

All this can be expressed in the language of thermogravity:

$$\Delta T\, dS = \left(T_{PG} - T_{\substack{test\\particle}}\right) dS = dM$$

To verify this, note that

$$T = \frac{1}{2\pi}\frac{dv}{dt} \;\; (\text{Unruh!}) \qquad S = \pi r^2 M_{pl}^2 = \frac{\pi r^2}{G} \qquad dM = 4\pi r^2 \rho\, dr$$

$$dS = 2\pi G^{-1} r\, dr$$

I do not claim to understand this curious result. But maybe it would be understandable to Eric Verlinde.

4 Oct 2011

V. Beyond Spherical Symmetry

In the context of this very simple example, we have seen how the PG "condensate", related to the flow parameter v in the PG metric, has a description distinct from that of a massive test particle released from rest at infinity. This distinction is related to the nontriviality of the lapse function in the related ADM description. However, left unanswered is the question of how to generalize this to less symmetrical geometries. Especially interesting would be to simply replace the continuum mass distribution of our hollow-sphere "Christmas tree ornament" with a realistic assemblage of atoms.

We found in the preceding sections that simplifications occur in the nonrelativistic limit. This suggests that rather than attacking the problem via the general ADM formalism, which looks pretty demanding to me, it may be a bit easier to go to the PPN description of perturbative general relativity. A literature search (via Google!) turned up a useful paper by Sotiriou and Barausse (arXiv 0612065). Their equation 29 expresses the GB density in terms of the Newtonian potential. Writing

$$\mathcal{L}_{GB} \propto \sqrt{-g}\; R_{\mu\nu}^{\ \ \alpha\beta}\, R_{\lambda\sigma}^{\ \ \gamma\delta}\, \epsilon_{\alpha\beta\gamma\delta}\, \epsilon^{\mu\nu\lambda\sigma}$$

one has (up to a normalization factor C , which will be eventually matched to the MM scenario)

$$\mathcal{L}_{GB} = C \sum_{i,j=1}^{3} \left[(\partial_i \partial_j \Phi)(\partial_i \partial_j \Phi) - (\partial_i \partial_i \Phi)(\partial_j \partial_j \Phi) \right]$$

Here Φ is the Newtonian potential occurring in the usual PPN formulation:

$$g_{00} = (1 + 2\Phi) \qquad , etc,$$

For a spherically symmetric source, and of course for weak fields and low velocities, the normalization is

$$\Phi \cong -\frac{GM}{r}$$

We can connect with GB by writing

$$\Phi = \frac{V^2}{2}$$

Therefore, for this case, the second term in the expression for g vanishes, and

$$\mathcal{L}_{GB} = C \sum_{ij} (\partial_i \partial_j V^2)(\partial_i \partial_j V^2) \equiv C \sum_{ij} (\partial_i \partial_j u)(\partial_i \partial_j u)$$

But

$$\partial_i \partial_j u(r) = \frac{x_i x_j}{r^2} u'' + \left(\delta_{ij} - \frac{x_i x_j}{r^2} \right) \frac{u'}{r}$$

This leads to the result

$$\mathcal{L}_{GB} = C \left[(u'')^2 + 2\left(\frac{u'}{r}\right)^2 \right] \sim \frac{G^2 M^2}{r^6}$$

From page 3 of these notes we have a different expression

$$\mathcal{L}_{GB} \sim \frac{1}{2r^2} \left(u u'' + (u')^2 \right) \sim \frac{G^2 M^2}{r^6}$$

Nevertheless, the bottom line result is the same.

The distinction between the formal structures becomes sharpened when spherical symmetry is abandoned. The PPN expression can be written as a total space divergence

$$\mathcal{L}_{GB} = C \sum_{ij} \partial_i \left[(\partial_j \Phi)(\partial_i \partial_j \Phi) - (\partial_i \Phi)(\partial_j \partial_j \Phi) \right]$$

$$- C \sum_{ij} \left[(\partial_j \Phi)(\partial_i \partial_i \partial_j \Phi) - (\partial_i \Phi)(\partial_i \partial_j \partial_j \Phi) \right]$$

$$= C \vec{\nabla} \cdot \left[\sum_j (\partial_j \Phi) \vec{\nabla} (\partial_j \Phi) - (\vec{\nabla} \Phi)(\nabla^2 \Phi) \right]$$

Then, in the spirit of the example of the "atomized Christmas tree ornament", we assume that the sources of gravity are highly localized, such as nuclei, protons, or neutron stars. Again, the second term in the above expression for $\mathcal{L}_{GB}$ vanishes. The first term can be written as

$$\mathcal{L}_{GB} = C\,\vec{\nabla}\cdot\left[\frac{1}{2}\vec{\nabla}\,|\vec{\nabla}\Phi|^2\right] = C\,\nabla^2\,\frac{|g|^2}{2} \equiv C\,\nabla^2 U$$

This suggests that the important source for the flow is not the gradient of the Newtonian potential, but rather the gradient of the "gravistatic energy density" $U = |g|^2/2$. (Actually U does not have those dimensions; it has dimensions of inverse second power of distance.) We now explore this a little more.

If we define a flow to be in the direction of the gradient of U, we can, by analogy with what we did for the PG discussion in MM Addendum, set up a small test volume in space which goes with the flow, as shown in the figure below. This time there will be a cross-sectional area $\Delta A(t)$ intercepting the flow lines, and a floor "a" and ceiling "b" each characterized as surfaces of constant U. (Fig. 1)

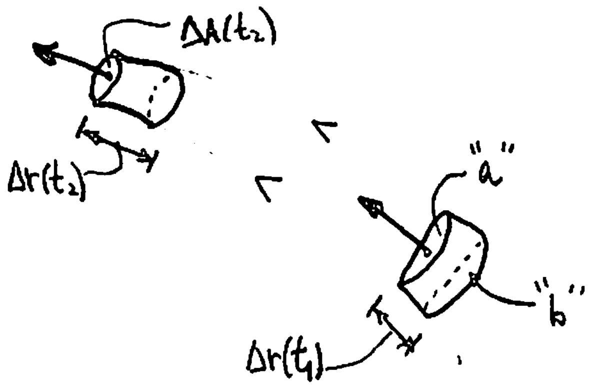


Fig. 1: Generalized Einstein elevator, now flowing "with the condensate" rather than inertially.

Given this construction, and the fact that only the surfaces a and b contribute, the GB space integral can be done:

$$L_{GB}(t) = C\int d^3x\,\mathcal{L}_{GB} = C\int d^2S\,\hat{n}\cdot\vec{\nabla}U = C\left[\left(\Delta A\frac{\partial U}{\partial s}\right)_b - \left(\Delta A\frac{\partial U}{\partial s}\right)_a\right] = C\Delta r\frac{\partial}{\partial s}\left(\Delta A\frac{\partial U}{\partial s}\right)$$

Thanks to the (approximately) stationary nature of the flow, the distance between floor "a" and ceiling "b" of the small test volume is $\Delta r = v(t)\Delta t$, with Δt independent of time. One finds

$$S_{GB} = C\int_{t_1}^{t_2} dt\,L_{GB}(t) = C\int_{t_1}^{t_2} dt\,v\,\Delta t\,\frac{\partial}{\partial s}\left(\Delta A\frac{\partial U}{\partial s}\right) = C\left[\Delta t\,\Delta A\frac{\partial U}{\partial s}\bigg|_{t_2} - \Delta t\,\Delta A\frac{\partial U}{\partial s}\bigg|_{t_1}\right]$$

As required on general grounds, this result is consistent with the requirement that the GB term is "topological":

$$S_{GB} = 2\pi \left[N(t_2) - N(t_1) \right]$$

And again we may obtain a "topological density" n(x) by dividing out the volume from the above expression. We obtain

$$n = \frac{C}{2\pi} \frac{\Delta t\, \Delta A}{\Delta r\, \Delta A} \frac{\partial U}{\partial s} = \frac{C}{2\pi v} \frac{\partial U}{\partial s} = \frac{C}{2\pi v^2} \left(\vec{v} \cdot \vec{\nabla} \right) U$$

It is interesting that these considerations do not determine in this generalized case what the magnitude of the flow should be. Perhaps this is only determined, if at all, at the level of a specific condensed-matter analogue model.

Meanwhile, we may obtain the normalization constant C . From p. 5 of the MM addendum,

$$n = \frac{M_{pl}^2}{16\pi^2 H^2} \left(\frac{v^2}{r^2} \frac{\partial v}{\partial r} \right)$$

For a simple compact source

$$v^2 = \frac{2GM}{r} \qquad \frac{v^2}{r^2} \frac{\partial v}{\partial r} = \frac{1}{2} \left(\frac{v^2}{r^2} \right)^{3/2}$$

Therefore

$$n = \frac{\sqrt{2}\, M_{pl}^2}{16\pi^2 H^2} \left(\frac{GM}{r^3} \right)^{3/2}$$

We can compare this with our new expression for the density n (x) :

$$n = \frac{C}{2\pi v} \frac{\partial}{\partial r} \left(\frac{g^2}{2} \right) = \frac{C}{4\pi v} \frac{\partial}{\partial r} \left(\frac{GM}{r^2} \right)^2 = \frac{C}{\pi} \sqrt{\frac{r}{2GM}} \frac{(GM)^2}{r^5} = \frac{C}{\pi\sqrt{2}} \left(\frac{GM}{r^3} \right)^{3/2}$$

We conclude that

$$C = \frac{M_{pl}^2}{8\pi H^2}$$