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Masses and Mixings of Quarks and Leptons

I. Introduction

For a long time Hong-Mo Chan and his collaborators have been studying the flavor problem via an approach involving a "rotating mass matrix". For the last year I too have worked on their approach. One of my outputs is a simple set of candidate formulae for masses and mixings of the quarks and leptons. In this note these results are summarized. No attempt will be made in justification of the formulae. The position here will be that the "derivations" of the formulae are less believable than the output results, and therefore not worth including.

II. Charged Leptons.

The input parameters are the mass of the tau lepton and a new, mysterious mass parameter m .

$$m \approx 7 \text{ MeV}$$

Define a mixing parameter theta, taken here to be real and positive:

$$\theta^4 \equiv \frac{m}{m_\tau}$$

Then, up to small corrections of higher order in theta, we assume the mass matrix is

$$M = \begin{pmatrix} \theta^4 & \theta^3 & 0 \\ \theta^3 & \theta^2 & \theta^2 \\ 0 & \theta^2 & 1 \end{pmatrix} m_\tau$$

It is diagonalized by a mixing matrix U , which is (up to small corrections of higher order in theta)

$$M = U M_{\text{Diag}} U^{-1}$$

$$U \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta^2 \\ 0 & -\theta^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & \theta & 0 \\ -\theta & 1 & \theta^2 \\ \theta^3 & -\theta^2 & 1 \end{pmatrix}$$

The diagonalized mass matrix is

$$M_{\text{Diag}} \cong \begin{pmatrix} -\theta^6 & 0 & 0 \\ 0 & \theta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_\tau$$

The masses of muon and electron are given by the formulae

$$m_\mu \cong \Theta^2 m_\tau = \sqrt{m_e m_\tau} \approx 110 \text{ MeV} \quad (106 \text{ MeV})$$

$$|m_e| \cong \Theta^6 m_\tau = \frac{m_e^2}{m_\mu} \approx 450 \text{ keV} \quad (511 \text{ keV})$$

(The experimental numbers from the PDG are given in parentheses.)

Hereafter we replace the mixing parameter θ with θ_q , in order to recycle the indexless θ for the quark sector.

III. Quarks

HMC assume the mixing mechanism is universal, i.e. independent of fermion species. We express this idea here, with two exceptions. The (recycled) mixing parameter, say, for the down quarks is assumed to be

$$\theta^4 \cong -\frac{m}{m_f}$$

This means that this time theta is complex, with its phase being one of the fourth roots of -1. With hindsight, this choice is also viable (and preferable!) for the charged leptons as well.

The mass matrix for the down quarks is assumed to be

$$M = \begin{pmatrix} Z|\theta|^4 & |\theta|^2\theta & 0 \\ |\theta|^2\theta^* & |\theta|^2 & |\theta|^2 \\ 0 & |\theta|^2 & 1 \end{pmatrix} m_f$$

The very small first-generation diagonal element has been tweaked a bit by the factor Z, assumed here not to be unity, but still of order unity. But other than introduction of this tweak, and introduction of the phase factor, the formal structure of the mass matrix is in fact the same as that for the charged leptons.

The diagonalization of this matrix is to good approximation identical to the previous case:

$$M = U m_{\text{Diag}} U^{-1}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & |\theta|^2 \\ 0 & -|\theta|^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta & 0 \\ -\theta^* & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & \theta & 0 \\ -\theta^* & 1 & |\theta|^2 \\ |\theta|^2\theta^* & -|\theta|^2 & 1 \end{pmatrix}$$

The diagonalized mass matrix m_{Diag} becomes

$$m_{\text{Diag}} \cong \begin{pmatrix} (z-1)|\theta|^4 & 0 & 0 \\ 0 & |\theta|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_t$$

This gives, for the masses of the strange and down quarks,

$$m_s = |\theta_d|^2 m_t \cong \sqrt{m m_t} \cong 165 \text{ MeV} \quad (104 \pm 30 \text{ MeV})$$

$$m_d = |(z_d-1)|\theta_d|^4 m_t \cong |(z_d-1)|m = |(z_d-1)|(7 \text{ MeV}) \quad (4.7 \pm 1.3 \text{ MeV})$$

Again we recycle the θ parameter with the notation-change $\theta \rightarrow \theta_d$.

The same procedure can be made for the up quark sector. We again assume the recycled theta parameter satisfies the relation

$$\theta^4 = -\left(\frac{m}{m_t}\right)$$

Evidently, after again recycling the θ parameter,

$$m_c = |\theta_u|^2 m_t = \sqrt{m m_t} \approx 1.1 \text{ GeV} \quad (1.3 \pm 0.1 \text{ GeV})$$

$$m_u = |(z_u-1)|\theta_u|^4 m_t = |(z_u-1)|m \approx |(z_u-1)|(7 \text{ MeV}) \quad (2.5 \pm 1 \text{ MeV})$$

All these results are in satisfactory agreement with the data.

IV. CKM Mixing

The CKM mixing matrix is constructed from the mixing matrices U exhibited above. We only need to be careful in appending subscripts on the sundry parameters:

$$V_{\text{CKM}} = U_u^{-1} U_d \cong \begin{pmatrix} 1 & -\theta_u & |\theta_d^2 \theta_u| \\ \theta_u & 1 & -|\theta_u|^2 \\ 0 & |\theta_u|^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta_d & 0 \\ -\theta_d^* & 1 & |\theta_d|^2 \\ |\theta_d|^2 \theta_d^* & -|\theta_d|^2 & 1 \end{pmatrix}$$

$$\cong \begin{pmatrix} 1 & (\theta_d - \theta_u) & -(|\theta_d|^2 - |\theta_u|^2)\theta_u \\ -(\theta_d - \theta_u)^* & 1 & (|\theta_d|^2 - |\theta_u|^2) \\ (|\theta_d|^2 - |\theta_u|^2)\theta_d^* & -(|\theta_d|^2 - |\theta_u|^2) & 1 \end{pmatrix}$$

$$V_{CKM} \approx \begin{pmatrix} 1 & (\theta_d - \theta_u) & -|\theta_d|^2 \theta_u \\ -(\theta_d - \theta_u) & 1 & |\theta_d|^2 \\ -|\theta_d|^2 \theta_d^* & -|\theta_d|^2 & 1 \end{pmatrix}$$

The magnitudes of the mixings involving the third generation are given by the formulae

$$|V_{cb}| \approx |\theta_d|^2 \approx \sqrt{\frac{m_c}{m_b}} \approx 0.04 \quad (.042)$$

$$|V_{ub}| \approx |\theta_d|^2 |\theta_u| \approx \sqrt{\frac{m_c}{m_b}} \approx 0.003 \quad (.0036)$$

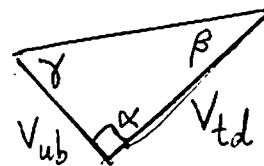
$$|V_{td}| \approx |\theta_d|^2 |\theta_d| \approx \sqrt{\frac{m_c}{m_b}} \approx 0.008 \quad (.0087)$$

These also agree very well with experiment.

We also see that the phases of the corner elements V_{ub} and V_{td} are each some choice of $\sqrt[4]{(-i)}$. This means that in the unitarity triangle the angle between them must be some multiple of 90 degrees.

Consequently, it is easy to choose the phases of V_{ub} and V_{td} to satisfy the last CKM experimental constraint:

$$\alpha = 90^\circ \quad (88^\circ \pm 6^\circ)$$



V. Neutrinos

I have no detailed description for the neutrino sector. If the seesaw mechanism is invoked, the structure of the neutrino mass matrix M_{Neutrino} is

$$M_{\text{Neutrino}} = M_{\text{Dirac}}^\dagger M_{\text{Majorana}}^{-1} M_{\text{Dirac}}$$

The Dirac part is what we expect to be subjected to the above rules for the mass matrix. Because large mixings seem to characterize the neutrino sector, it would appear that the parameter theta in this case should be order 1. This would seem to imply that the typical Dirac mass eigenvalue should be of order m , namely of order 10 MeV.

In any case, it is likely that the "tweaks" encountered in the quark sector will be magnified when the mixing angles are so large. Nevertheless, it seems worth trying to put some better numbers into the conjectures contained in the previous paragraph. In this spirit, we make some very simple assumptions:

- 1) We assume the Dirac mass matrix to be real and symmetric.
- 2) We assume the Majorana mass matrix to be a multiple of the identity.

This guarantees that, up to small corrections coming from charged-lepton mixing, the MNS matrix is the same matrix as what diagonalizes the Dirac mass matrix of the neutrinos. So we can compare it with the form we have assumed for the other mixing matrices. The MNS matrix, to good approximation, takes a tribimaximal form. With an appropriate choice of phases, it is

$$U = U_{\text{MNS}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Write the diagonalized Dirac matrix m_{Diag} as follows:

$$m_{\text{Diag}} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} m \quad (m = 7 \text{ MeV})$$

The undiagonalized Dirac matrix can then be reconstructed, and is given by

$$M_{\text{Dirac}} = \begin{pmatrix} \frac{2E_1 + E_2}{3} & \frac{E_2 - E_1}{3} & \frac{E_1 - E_2}{3} \\ \frac{E_2 - E_1}{3} & \frac{3E_3 + 2E_2 + E_1}{6} & \frac{3E_3 - 2E_2 - E_1}{6} \\ \frac{E_1 - E_2}{3} & \frac{3E_3 - 2E_2 - E_1}{6} & \frac{3E_3 + 2E_2 + E_1}{6} \end{pmatrix} m$$

From this point onward, there are many avenues open for exploration. Here we choose the normal hierarchy, with the three neutrino masses given by

$$m_{\nu_3} = \frac{m_3^2}{M} = 50 \text{ meV} \quad m_{\nu_2} = \frac{m_2^2}{M} = 8 \text{ meV} \quad m_{\nu_1} = \frac{m_1^2}{M} = 2 \text{ meV}$$

The choice of first-generation mass is of course just a guess. From these choices we infer

$$\frac{m_1}{m_3} = \frac{\epsilon_1}{\epsilon_3} = \pm 0.2 \qquad \frac{m_2}{m_3} = \frac{\epsilon_2}{\epsilon_3} = 0.4$$

We leave open the choice of sign for m_1 . The reconstructed off-diagonal Dirac mass matrix is then

$$M_{\text{Dirac}} = \begin{pmatrix} .20(0) & .07(.20) & -.07(-.20) \\ .07(.20) & .67(.51) & .33(.13) \\ -.07(-.20) & .33(.13) & .67(.51) \end{pmatrix} \epsilon_3 m$$

The values in parentheses are appropriate for the choice of negative sign for m_1 .

This form, as well as a broad class of generalizations, shows that it will require major "tweaks" in order to connect its structure smoothly to the small- θ form we have assumed for the quarks and for the charged leptons. Exploration of this question is beyond the scope of this note.

Nevertheless, we may still infer that the appropriate mass range for the heaviest neutrino is

$$50 \text{ meV} < m_{\nu_3} < 500 \text{ meV}$$

The lower limit comes from data on neutrino oscillations and the upper limit from astrophysical bounds on the sum of the masses.

Because of the large mixings, the natural guess for the largest Dirac mass is somewhere between a lower limit of 7 MeV and an upper limit of, say, 20 MeV. This allows an estimate of the scale of the heavy Majorana neutrino masses. With $m_{\nu_3} = 200 \text{ meV}$ and $m_3 = 14 \text{ MeV}$, the seesaw formula gives

$$M_{\text{Majorana}} \approx \frac{M_{\text{Dirac}}^2}{m_3} \approx \frac{(14 \text{ MeV})^2}{(200 \text{ meV})} \approx 10^3 \text{ TeV}$$

This estimate is uncertain to at least an order of magnitude. But even so, we find that the Majorana mass scale is still very small compared to conventional wisdom.