Math 1497 - Calc 2

Techniques of integration

1. Substitution

ex $\displaystyle\int \frac{x^3}{x^2+1}\,dx$ let $u = x^2+1$

$du = 2x\,dx$

$$\int \frac{x^2 \cdot x\,dx}{x^2+1} = \int \frac{x^2 \frac{du}{2}}{u} = \frac{1}{2}\int \frac{x^2\,du}{u}$$

use sub again

$$= \frac{1}{2}\int \frac{u-1}{u}\,du$$

$$= \frac{1}{2}\int 1-\frac{1}{u}\,du = \frac{1}{2}\left(u - \ln|u|\right) + C$$

$$= \frac{1}{2}\left(x^2+1 - \ln|x^2+1|\right) + C$$

could absorb $\frac{1}{2}$ into C

2. Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Ex $\int x e^{-2x}\,dx$ $u = x$ $v = \frac{e^{-2x}}{-2}$

$du = dx$ $dv = e^{-2x}\,dx$

$$= -\frac{x e^{-2x}}{2} - \int \frac{e^{-2x}}{-2}\,dx$$

$$= -\frac{x e^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= -\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + c$$

Now we consider a series of "trig" integrals

Form

$$\int \sin^m x \cos^n x\,dx, \quad \int \tan^m x \sec^n x\,dx$$

the basic idea here is to introduce a substitution to reduce the integrals to polynomial integrals

we first consider

$$\int \sin^m x \cdot \cos^n x \, dx$$

where m & n are integers.

possibilities
(i) m even n odd
(ii) m odd n even
(iii) m, n odd
(iv) m, n even

we consider these by example

Ex 1 $\int \underbrace{\sin^2 x}_{\text{most complicated}} \cos x \, dx$

if $u = \sin x$ and $\int u^2 du = \frac{u^3}{3} + c$

$du = \cos x \, dx$

so $= \frac{1}{3} \sin^3 x + c$

Ex 2 $\int \sin^2 x \cos^3 x \, dx$

Note $\int \sin^2 x \cos^2 x \cos x \, dx$

$u = \sin x$ so $\int u^2 \cos^2 x \, du$

$du = \cos x \, dx$

but $\cos^2 x = 1 - \sin^2 x = 1 - u^2$

So $\int u^2 (1 - u^2) \, du$

$\int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + c$

$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + c$

II m odd n even

$$\int \sin^3 x \cos^4 x \, dx$$

$$\int \sin^4 x \cos^4 x \sin x \, dx$$ let $u = \cos x$, $du = -\sin x \, dx$

$$\int \sin^4 x \, u^4 (-du)$$

$\uparrow$?

$$\sin^4 x = (\sin^2 x)^2 = (1-\cos^2 x)^2$$ Now use sub

$$-\int (1-u^2)^2 u^4 \, du$$ $(1-u^2)^2 = 1-2u^2+u^4$

$$-\int u^4 - 2u^6 + u^8 \, du = -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C$$

$$= -\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C$$

III both odd

$$\int \sin^3 x \cos^5 x \, dx$$

pull off $\cos x$

$$\int \sin^2 x \cos^5 x \sin x \, dx \qquad \int \sin^3 x \cos^4 x \cos x \, dx$$

$u = \cos x$ $\qquad$ $u = \sin x$

$du = -\sin x \, dx$ $\qquad$ $du = \cos x \, dx$

$\sin^2 x = 1 - \cos^2 x$ $\qquad$ $\cos^2 x = 1 - \sin^2 x$

$$-\int (1-u^2) u^5 \, du \qquad \int u^3 (1-u^2)^2 \, du$$

this one is easier

$$-\frac{u^6}{6} + \frac{u^8}{8} + C = -\frac{1}{6}\cos^6 x + \frac{1}{8}\cos^8 x + C$$

IV Both even

Ex $\int \sin^2 x \, dx$ or $\int \cos^2 x \, dx$

Need trig identities

$$\cos 2x = 1 - 2\sin^2 x \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= 2\cos^2 x - 1 \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

So $\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx = \frac{1}{2} \int 1 - \cos 2x \, dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2} \right] + C$

$\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx = \frac{1}{2} \int 1 + \cos 2x \, dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2} \right] + C$

Ex $\int \sin^2 x \cos^2 x \, dx$

$$\int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} \, dx$$

$$\frac{1}{4}\int (1-\cos^2 2x)\, dx \leftarrow$$ need identity again

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$\frac{1}{4}\int \left(1 - \frac{1+\cos 4x}{2}\right) dx$$

$$\frac{1}{4}\int \frac{1}{2} - \frac{\cos 4x}{2} \, dx = \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + C$$