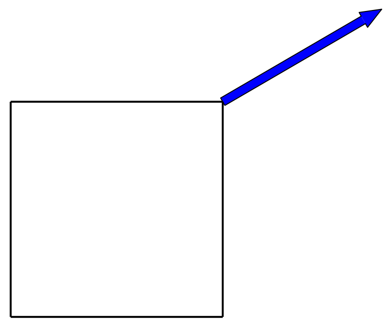
Calculus 3 - Line Integrals Over Vector Fields

Consider the following. Suppose we have a 1 kg block sitting on the ground. We apply a force of say 10 N to the corner of the block where the direction of the force is 30^o to the horizon (see the attached figure). If we assume we are on a frictionless surface, what force would be done moving the block 2 m.



From elementary physics we have

$$W = Fd\cos\theta \tag{1}$$

Here $F = 10$, $d = 2$, and $\theta = 30$ and from (1) we obtain

$$W = 10 \times 2 \times \cos 30 = 10\sqrt{3}Nm. \tag{2}$$

In this problem, we have assume that the force applied was constant and in the same direction (*i.e.* 30^o). Now we assume all these change. We will assume:

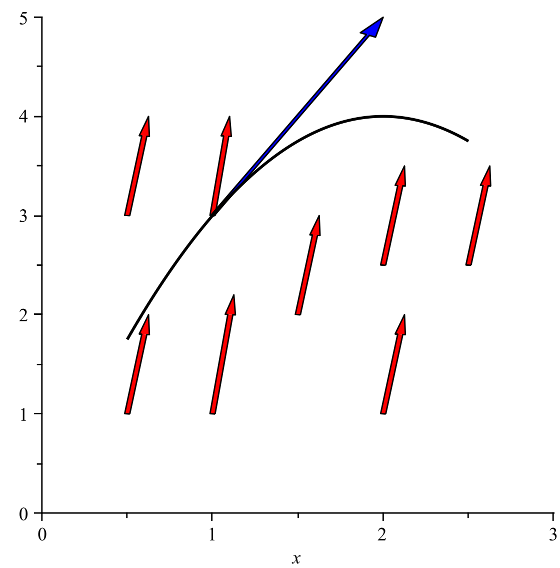
1. that the force's direction changes and depends on position so the force is now $\vec{F}(x, y)$ (a vector)

2. the path we follow will depend on both x and y and so we will move along some curve C.

We assume that we move along a curve C whose position can be represented by a vector

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

and we are moving through a vector field $\vec{F}$



To calculate the work, we project $\vec{F}$ onto the tangent vector $\vec{T}$, multiply by a small distance ds and integrate over the entire curve C, *i.e.*,

$$W = \int_C \vec{F} \cdot \vec{T} ds. \tag{3}$$

From our section on vector functions we have

$$\vec{T} = \frac{\vec{r}\,'}{\|\vec{r}\,'\|} = \frac{\langle x', y' \rangle}{\sqrt{x'^2 + y'^2}},$$

and from arc length that ds is given by

$$ds = \sqrt{x'^2 + y'^2}\, dt.$$

Now suppose the $\vec{F}$ is given by

$$\vec{F} = \langle P(x,y), Q(x,y) \rangle,$$

then from (3) we obtain

$$\begin{aligned} W &= \int_C \vec{F} \cdot \vec{T} ds \\ &= \int_C \langle P(x,y), Q(x,y) \rangle \cdot \frac{\langle x', y' \rangle}{\sqrt{x'^2 + y'^2}} \sqrt{x'^2 + y'^2} dt \\ &= \int_C \langle P(x,y), Q(x,y) \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt \\ &= \int_C \langle P(x,y), Q(x,y) \rangle \cdot \langle dx, dy \rangle \\ &= \int_C P\,dx + Q\,dy \end{aligned} \tag{4}$$

so we define the line integral along the curve C over the vector field $\vec{F}$ as

$$\int_C P\,dx + Q\,dy \tag{5}$$

Example 1. Evaluate

$$\int_C ydx + 2xdy \tag{6}$$

where C is the curve $y = x^2$ from $(1,1)$ to $(2,4)$.

Soln.

Since $y = x^2$, then $dy = 2x\,dx$ and our line integral becomes

$$\int_1^2 x^2 dx + 2x \cdot 2x dx = \int_1^2 5x^2\,dx = \frac{5}{3}x^3\Big|_1^2 = \frac{35}{3}. \tag{7}$$

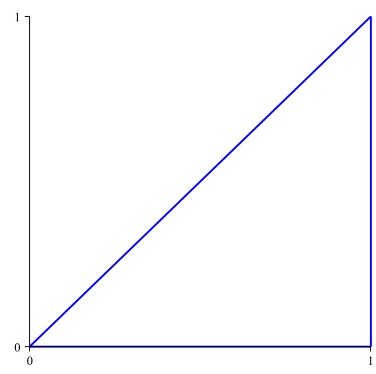
Example 2. Evaluate

$$\int_C x dx - xy dy \tag{8}$$

where C is the curve $y = 0$, $x = 1$ and $y = x$ going around the curve counterclockwise.

Soln.

We first draw the picture



As there are three distinct curves, we will have three line integrals

C_1 $-$ The line $y = 0$ from $x = 0 \to 1$. So $dy = 0$ and from (8)

$$\int_0^1 x dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2} \tag{9}$$

C_2 $-$ The line $x = 1$ from $y = 0 \to 1$. So $dx = 0$ and from (8)

$$\int_0^1 -y dy = -\frac{1}{2}y^2\Big|_0^1 = -\frac{1}{2} \tag{10}$$

C_3 − The line $y = x$ from $x = 1 \to 0$. So $dy = dx$ and from (8)

$$\int_1^0 xdx - x^2dx = \frac{1}{2}x^2 - \frac{1}{3}x^3\Big|_1^0 = -\frac{1}{6} \tag{11}$$

So

$$\int_C xdx - xydy = \frac{1}{2} - \frac{1}{2} - \frac{1}{6} = -\frac{1}{6} \tag{12}$$

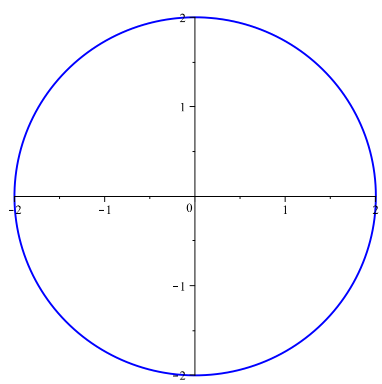
Example 3. Evaluate

$$\int_C xdy \tag{13}$$

where C is the curve $x^2 + y^2 = 4$ going around the curve counterclockwise.

Soln.

We first draw the picture



We certainly could solve for y but it's really much easier to parameterize the curve. Here

$$x = 2\cos t, \quad y = 2\sin t, \quad t : 0 \to 2\pi \tag{14}$$

so

$$dx = -2\sin t\, dt, \quad dy = 2\cos t\, dt, \tag{15}$$

and (13) becomes

$$\begin{aligned}
\int_0^{2\pi} 2\cos t\,(2\cos t)\; dt &= 4\int_0^{2\pi} \cos^2 t\,dt \\
&= 4\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt \\
&= 2\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{2\pi} \\
&= 4\pi
\end{aligned} \tag{16}$$

Example 4. Evaluate

$$\int\limits_C \mathbf{F}\cdot\mathbf{dr} \tag{17}$$

where $\mathbf{F}(x,y) = xy\mathbf{i} + 3y^2\mathbf{j}$ and $C : \mathbf{r}(t) = 11t^4\mathbf{i} + t^3\,\mathbf{j} \quad 0 \le t \le 1$

Soln.

I am going to turn this into a problem I'm more familiar with. here, we identify that

$$P = xy, \quad Q = 3y^2 \tag{18}$$

so (17) is

$$\int\limits_C xydx + 3y^2dy. \tag{19}$$

Next the curve, so

$$x = 11t^4, \quad y = t^3 \tag{20}$$

the curve is already parameterized. So

$$dx = 44t^3\,dt, \quad dy = 3t^2\,dt, \tag{21}$$

and (19) becomes

$$\begin{aligned}\int_0^1 11t^3 \cdot t^3 \cdot 44t^3\,dt + 3t^6 \cdot 3t^2\,dt &= \int_0^1 \left(484t^{10} + 9t^8\right)\,dt \\ &= 44t^{11} + t^9\Big|_0^1 \\ &= 45\end{aligned} \tag{22}$$

3D Line Integrals over Vector Fields

If the vector field is

$$\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle, \tag{23}$$

then the 3D line integral is

$$\int\limits_C P\,dx + Q\,dy + R\,dz \tag{24}$$

Example 5. Evaluate

$$\int\limits_C ydx + zdy + xdz \tag{25}$$

where C is the line joining $(1,0,1)$ to $(4,1,2)$

Soln.

The line is

$$x = 1 + 3t, \quad y = t, \quad z = 1 + t,$$

and so

$$dx = 3dt \quad dy = dt, \quad dz = dt$$

$$\int_0^1 t3dt + (1+t)dt + (1+3t)dt = \int_0^1 (2+7t)\,dt = \frac{11}{2}. \tag{26}$$