C3 Trigonometry

1a) $\dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$

$$1 + \tan^2\theta = \sec^2\theta$$
$$\tan^2\theta = \sec^2\theta - 1$$

b) $2\tan^2\theta + 4\sec\theta + \sec^2\theta = 2$

$$2(\sec^2\theta - 1) + 4\sec\theta + \sec^2\theta = 2$$
$$2\sec^2\theta - 2 + 4\sec\theta + \sec^2\theta = 2$$
$$3\sec^2\theta + 4\sec\theta - 4 = 0$$
$$(3\sec\theta - 2)(\sec\theta + 2) = 0$$

$\sec\theta = 2/3$ $\quad$ $\sec\theta = -2$

$\cos\theta = 3/2$ $\quad$ $\cos\theta = -1/2$

$\therefore$ NO SOLUTION $\quad$ $\theta = \cos^{-1}(-1/2)$

$\theta = 120, 240$

2a) $\sin 2x$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(x+x) = \sin x\cos x + \cos x\sin x$$
$$\sin 2x = 2\sin x\cos x$$

b) $\operatorname{cosec} x - 8\cos x = 0$

$$\frac{1}{\sin x} - 8\cos x = 0$$

$$1 - 8\sin x\cos x = 0$$
$$1 - 4\sin 2x = 0$$
$$\sin 2x = 1/4$$
$$2x = \sin^{-1}(1/4)$$
$$= 0.2526802551,\ 2.88891239$$
$$x = 0.13,\ \cancel{0}\ 1.44$$

3) $x = \cos(2y + \pi)$

$$\frac{dx}{dy} = -2\sin(2y + \pi)$$

when $y = \pi/4$

$$\frac{dx}{dy} = -2\sin\left(\frac{2\pi}{4} + \pi\right)$$

$$= 2$$

$$\therefore \frac{dy}{dx} = 1/2$$

$$y = mx + c$$

$$\pi/4 = 1/2(0) + c$$

$$c = \pi/4$$

$$y = 1/2\, x + \pi/4$$

4a i) $\sin(2\theta + \theta) = (\sin 2\theta)\cos\theta + (\cos 2\theta)\sin\theta$

$$= (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$$

$$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$$

$$= 3\sin\theta(\cos^2\theta) - \sin^3\theta$$

$$= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

ii) $8\sin^3\theta - 6\sin\theta + 1 = 0$

$$1 = 6\sin\theta - 8\sin^3\theta$$

$$1 = 2\sin(3\theta)$$

$$1/2 = \sin(3\theta)$$

$$3\theta = \sin^{-1}(1/2)$$

$$3\theta = 1/6\,\pi,\ 5/6\,\pi$$

$$\theta = 1/18\,\pi,\ 5/18\,\pi$$

b) $\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$

$\sin(45-30) = \sin45\cos30 - \cos45\sin30$

$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}$

$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$= \frac{1}{4}(\sqrt{6} - \sqrt{2})$

5a) $\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

b) $2\cot^2\theta - 9\operatorname{cosec}\theta = 3$

$2(\operatorname{cosec}^2\theta - 1) - 9\operatorname{cosec}\theta = 3$

$2\operatorname{cosec}^2\theta - 2 - 9\operatorname{cosec}\theta = 3$

$2\operatorname{cosec}^2\theta - 9\operatorname{cosec}\theta - 5 = 0$

$(2\operatorname{cosec}\theta + 1)(\operatorname{cosec}\theta - 5) = 0$

$\operatorname{cosec}\theta = -\frac{1}{2}$ $\quad$ $\operatorname{cosec}\theta = 5$

$\sin\theta = -2$ $\quad$ $\sin\theta = \frac{1}{5}$

NO SOLUTION $\quad$ $\theta = 11.53695903,$

168.4630497

$\underline{\theta = 11.5^\circ, 168.5^\circ}$

6a) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(2x+x) = (\cos2x)\cos x - (\sin2x)\sin x$

$= (\cos^2x - \sin^2x)\cos x - (2\sin x\cos x)\sin x$

$= \cos^3x - \sin^2x\cos x - 2\sin^2x\cos x$

$= \cos^3x - 3(\sin^2x)\cos x$

$= \cos^3x - 3\cos x(1 - \cos^2x)$

$= \cos^3x - 3\cos x + 3\cos^3x$

$= 4\cos^3x - 3\cos x$

6bi) $\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} \equiv 2\sec x$

$$\frac{\cos^2 x}{\cos x(1+\sin x)} + \frac{(1+\sin x)^2}{\cos x(1+\sin x)}$$

$$\frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{\cos x(1+\sin x)}$$

$$\frac{(\cos^2 x + \sin^2 x) + 1 + 2\sin x}{\cos x(1+\sin x)}$$

$$\frac{(1) + 1 + 2\sin x}{\cos x(1+\sin x)}$$

$$\frac{2 + 2\sin x}{\cos x(1+\sin x)}$$

$$\frac{2\cancel{(1+\sin x)}}{\cos x\cancel{(1+\sin x)}}$$

$$\frac{2}{\cos x}$$

$$2\sec x$$

ii) $2\sec x = 4$

$\sec x = 2$

$\cos x = 1/2$

$x = \pi/3, \frac{5\pi}{3}$

7a) $\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$\tan^2\theta + 1 = \sec^2\theta$

b) $2(\tan^2\theta) + \sec\theta = 1$

$2(\sec^2\theta - 1) + \sec\theta = 1$

$2\sec^2\theta - 2 + \sec\theta = 1$

$2\sec^2\theta + \sec\theta - 3 = 0$

$(2\sec\theta + 3)(\sec\theta - 1) = 0$

$\sec\theta = -\frac{3}{2}$ $\quad \sec\theta = 1$

$\cos\theta = -\frac{2}{3}$ $\quad \cos\theta = 1$

$\theta = 131.8$ $\quad \theta = 0, 360$

228.2

$\underline{\theta = 0^\circ, 131.8^\circ, 228.2^\circ}$

8a) See Q.4

b) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$= 3\left(\frac{\sqrt{3}}{4}\right) - 4\left(\frac{\sqrt{3}}{4}\right)^3$

$= \frac{3\sqrt{3}}{4} - 4\left(\frac{3\sqrt{3}}{64}\right)$

$= \frac{3\sqrt{3}}{4} - \frac{3\sqrt{3}}{16}$

$= \frac{12\sqrt{3}}{16} - \frac{3\sqrt{3}}{16}$

$= \frac{9\sqrt{3}}{16}$

9a) $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} \equiv 2\operatorname{cosec} 2\theta$

$$\frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

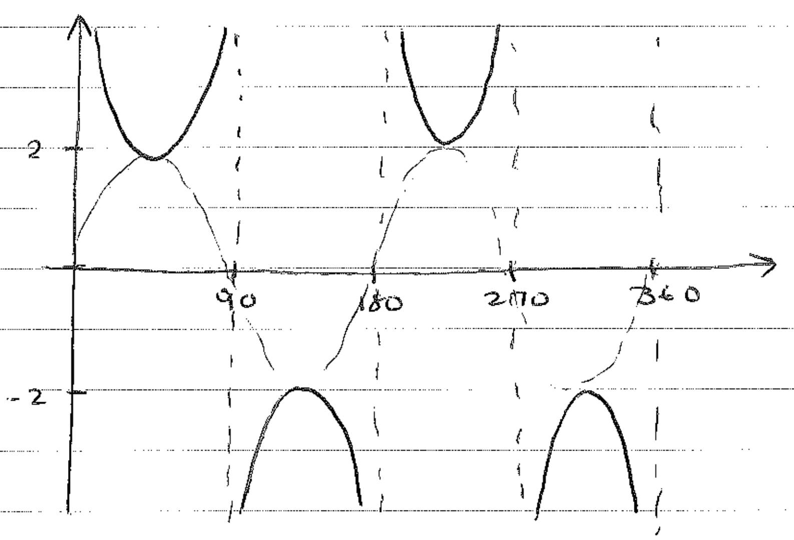
$$\frac{1}{\sin\theta\cos\theta}$$

$$\frac{2}{2\sin\theta\cos\theta}$$

$$\frac{2}{\sin 2\theta}$$

$$\underline{2\operatorname{cosec} 2\theta}$$

b)



c) $2\operatorname{cosec} 2\theta = 3$

$\operatorname{cosec} 2\theta = 3/2$

$\sin 2\theta = 2/3$

$2\theta = 41.8103149,\ 138.1896851,$

$401.8103149,\ 498.1896851$

$\theta = 20.9°,\ 69.1°,\ 200.9°,\ 249.1°$

10 $(\sec^2 x) - (\operatorname{cosec}^2 x) \equiv \tan^2 x - \cot^2 x$

$$\boxed{\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ 1 + \tan^2 x &= \sec^2 x \\ \cot^2 x + 1 &= \operatorname{cosec}^2 x \end{aligned}}$$

$(1 + \tan^2 x) - (\cot^2 x + 1)$

$1 + \tan^2 x - \cot^2 x - 1$

$\tan^2 x - \cot^2 x$

ii a) $y = \cos^{-1} x$

$90 - y = \sin^{-1} x$

b) $\cos^{-1} x + \sin^{-1} x = y + 90 - y$

$= 90$

$= \underline{\pi/2}$

11a) $\dfrac{\sin^2\theta}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}$

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$1 = \operatorname{cosec}^2\theta - \cot^2\theta$

b) $\operatorname{cosec}^4\theta - \cot^4\theta$

$(\operatorname{cosec}^2\theta + \cot^2\theta)(\operatorname{cosec}^2\theta - \cot^2\theta)$

$(\operatorname{cosec}^2\theta + \cot^2\theta)\ (1)$

$\operatorname{cosec}^2\theta + \cot^2\theta$

c) $\operatorname{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta$

$(\operatorname{cosec}^2\theta) + \cot^2\theta = 2 - \cot\theta$

$1 + \cot^2\theta + \cot^2\theta = 2 - \cot\theta$

$2\cot^2\theta + \cot\theta - 1 = 0$

$(2\cot\theta - 1)(\cot\theta + 1) = 0$

$\cot \theta = 1/2 \qquad \cot \theta = -1$

$\tan \theta = 2 \qquad \tan \theta = -1$

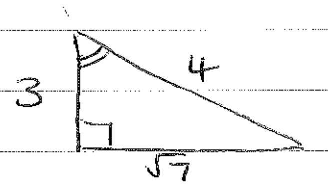
$\theta = 63.43494882, \quad \theta = -45, 135$

243.43494882

$\underline{\theta = 135^\circ}$

12a) $\sin 2A = 2 \sin A \cos A$

$\cos A = 3/4$



$\sqrt{4^2 - 3^2} = \sqrt{7} \qquad \sin A = -\sqrt{7}/4$

(sin is negative between 270 and 360)

$\sin 2A = 2 \cdot \frac{-\sqrt{7}}{4} \cdot \frac{3}{4}$

$= -\frac{6\sqrt{7}}{16}$

$= -\frac{3\sqrt{7}}{8}$

b i/ $\cos(2x + \pi/3) + \cos(2x - \pi/3) = \cos 2x$

$\cos 2x \cos \pi/3 \cancel{- \sin 2x \sin \pi/3} + \cos 2x \cos \pi/3 \cancel{+ \sin 2x \sin \pi/3}$

$2 \cos 2x (\cos \pi/3)$

$2 \cos 2x (1/2)$

$\underline{\cos 2x}$

ii/ $y = 3(\sin x)^2 + \cos 2x$

$\frac{dy}{dx} = 6 \sin x \cos x - 2 \sin 2x$

$= 3 \sin 2x - 2 \sin 2x$

$= \underline{\sin 2x}$

13a i)

$$\frac{\cos 2x}{\cos x + \sin x}$$

$$\frac{\cos^2 x - \sin^2 x}{\cos x + \sin x}$$

$$\frac{(\cancel{\cos x + \sin x})(\cos x - \sin x)}{(\cancel{\cos x + \sin x})}$$

$$\cos x - \sin x$$

ii)

$$\frac{1}{2}((\cos 2x) - \sin 2x) = \cos^2 x - \cos x \sin x - \tfrac{1}{2}$$

$$\frac{1}{2}((\cos^2 x - \sin^2 x) - \sin 2x)$$

$$\frac{1}{2}(\cos^2 x - \sin^2 x - (2\cos x \sin x))$$

$$\frac{1}{2}(\cos^2 x - (1 - \cos^2 x) - 2\cos x \sin x)$$

$$\frac{1}{2}(\cos^2 x - 1 + \cos^2 x - 2\cos x \sin x)$$

$$\frac{1}{2}(2\cos^2 x - 2\cos x \sin x - 1)$$

$$\cos^2 x - \cos x \sin x - \tfrac{1}{2}$$

b)

$$\cos\theta\left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) = \frac{1}{2}$$

$$\cos\theta(\cos\theta - \sin\theta) = \tfrac{1}{2}$$

$$\cos^2\theta - \cos\theta\sin\theta = \tfrac{1}{2}$$

$$\cos^2\theta - \cos\theta\sin\theta - \tfrac{1}{2} = 0$$

$$\tfrac{1}{2}(\cos 2\theta - \sin 2\theta) = 0$$

$$\cos 2\theta - \sin 2\theta = 0$$

$$\cos 2\theta = \sin 2\theta$$

$$\cancel{\tan 2\theta} =$$

c)

$$\cos 2\theta = \sin 2\theta$$

$$\tan 2\theta = 1$$

$$2\theta = \tan^{-1}(1)$$

$$= \pi/4,\ 5\pi/4,\ \frac{9\pi}{4},\ \frac{13\pi}{4}$$

$$\theta = \pi/8,\ 5\pi/8,\ \frac{9\pi}{8},\ \frac{13\pi}{8}$$

14a) $y = \sqrt{3}\cos x + \sin x$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\sin\alpha = \sqrt{3}$
$R\cos\alpha = 1$

$\tan\alpha = \sqrt{3}$ $\quad R^2 = (\sqrt{3})^2 + (1)^2$
$\alpha = \frac{1}{3}\pi$ $\quad R^2 = 4$
$R = 2$

$2\sin(x + \frac{1}{3}\pi)$

b) $2\sin(x + \frac{1}{3}\pi) = 1$
$\sin(x + \frac{1}{3}\pi) = \frac{1}{2}$
$x + \frac{1}{3}\pi = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{13}{6}\pi$
$\underline{x = \frac{1}{2}\pi, \frac{11}{6}\pi}$

15a) $y = 3\sin 2x + 4\cos 2x$
$\frac{dy}{dx} = 6\cos 2x - 8\sin 2x$

when $x=0$ $\quad \frac{dy}{dx} = 6$

gradient or normal $= -\frac{1}{6}$

$y = -\frac{1}{6}x + c$
$4 = -\frac{1}{6}(0) + c$
$c = 4$
$\underline{y = -\frac{1}{6}x + 4}$

b) $R\sin(2x+\alpha)$

$y = 3\sin 2x + 4\cos 2x$

$R\sin(2x+\alpha) = R\sin 2x\cos\alpha + R\cos 2x\sin\alpha$

$R\cos\alpha = 3$

$R\sin\alpha = 4$

$R = 5$ $\quad$ $\tan\alpha = 4/3$

$\alpha = \tan^{-1}(4/3)$

$\alpha = 0.927$ (3sf)

$\cancel{R\sin}$ $\underline{5\sin(2x+0.927)}$

c) $5\sin(2x+0.927) = 0$

$\sin(2x+0.927) = 0$

$2x+0.927 = \sin^{-1}(0)$

$2x+0.927 = 0, \pi, -\pi, 2\pi, \cancel{3\pi}$

$x = -0.46, 1.11, -2.03, 2.68$

$(-2.03, 0)$ $(-0.46, 0)$ $(1.11, 0)$ $(2.68, 0)$

16 $f(x) = 12\cos x - 4\sin x$

$R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\cos\alpha = 12$

$R\sin\alpha = 4$

$\tan\alpha = 4/12$

$R = \sqrt{12^2+4^2}$ $\quad$ $\alpha = \underline{18.43494882}$

$= \underline{4\sqrt{10}}$

b) $4\sqrt{10}\cos(x + 18.43494882) = 7$

$\cos(x + 18.43494882) = \dfrac{7}{4\sqrt{10}}$

$x + 18.43494882 = \cos^{-1}\left(\dfrac{7}{4\sqrt{10}}\right)$

$= 56.29951564,$

303.60048436

$x = 38.0, 285.2$

c) i) $-4\sqrt{10}$

ii) $\cos(x + 18.434494882) = -1$

$x + 18.434494882 = 180$

$x = 161.57^\circ$

17a) $3\cos\theta + 4\sin\theta$

$R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R = 5$ $\quad R\cos\alpha = 3$

$R\sin\alpha = 4$

$\tan\alpha = 4/3$

$\alpha = 53.13010235$

$5\cos(\theta - 53.13010235)$

b) Maximum value $= 5$

$5\cos(\theta - 53.13010235) = 5$

$\cos(\theta - 53.13010235) = 1$

$\theta - 53.13010235 = 0$

$\theta = 53.1$ (1dp)

c) $f(t) = 10 + 3\cos(15t) + 4\sin(15t)$

$= 10 + 5\cos(15t - 53.13010235)$

min temperature 5°

d) $\cos(15t - 53.10301023 5) = -1$

$15t - 53.10301023 5 = 180$

$15t = 233.10301023 5$

$t = 15.5$ (1dp)

18 $f(x) = 5\cos x + 12\sin x$

$R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\cos\alpha = 5$

$R\sin\alpha = 12$

$\underline{R = 13}$ $\tan\alpha = 12/5$

$\underline{\alpha = 1.176}$ (3dp)

b) $13\cos(x - 1.176) = 6$

$\cos(x - 1.176) = 6/13$

$x - 1.176 = 1.091067689,\ 5.192117618$

-1.091067689

$x = 0.0849323108 8,$

2.267067689

$x = 0.08,\ 2.27$ (2dp)

c i) $+13$

ii) $x = 1.176$

19 a) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A+A) = \cos A \cos A - \sin A \sin A$

$= (\cos^2 A) - \sin^2 A$

$= (1 - \sin^2 A) - \sin^2 A$

$= 1 - 2\sin^2 A$

b) $y = 3 \sin 2x$

$y = 4 \sin^2 x - 2\cos 2x$

$3 \sin 2x = 4 \sin^2 x - 2 \cos 2x$

$3 \sin 2x = 2(2 \sin^2 x) - 2\cos 2x$

$3 \sin 2x = 2(1 - \cos 2x) - 2\cos 2x$

$3 \sin 2x = 2 - 2\cos 2x - 2\cos 2x$

$3 \sin 2x = 2 - 4 \cos 2x$

$4\cos 2x + 3\sin 2x = 2$

c) $R \cos(2x - \alpha) = R\cos 2x \cos\alpha + R \sin 2x \sin\alpha$

$R \operatorname{cod} \alpha = 4$

$R \sin \alpha = 3$

$\tan \alpha = 3/4$

$R = 5$ $\qquad \alpha = 36.87$

$5 \cos(2x - 36.87)$

d) $5 \cos(2x - 36.87) = 2$

$\cos(2x - 36.87) = 2/5$

$2x - 36.87 = 66.42182152$

293.5781785

$x = 51.6°, 165.2°$

20a) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A+A) = \cos A \cos A - \sin A \sin A$

$= (\cos^2 A) - \sin^2 A$

$= (1 - \sin^2 A) - \sin^2 A$

$= 1 - 2\sin^2 A$

b) $2(\sin 2\theta) - 3(\cos 2\theta) - 3\sin\theta + 3 \equiv \sin\theta(4\cos\theta + 6\sin\theta - 3)$

$2(2\sin\theta\cos\theta) - 3(1-2\sin^2\theta) - 3\sin\theta + 3 =$

$4\sin\theta\cos\theta - 3 + 6\sin^2\theta - 3\sin\theta + 3$

$4\sin\theta\cos\theta + 6\sin^2\theta - 3\sin\theta$

$\sin\theta(4\cos\theta + 6\sin\theta - 3)$

c) $4\cos\theta + 6\sin\theta$

$R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$R\cos\alpha = 6$

$R\sin\alpha = 4$

$\tan\alpha = 4/6$ $\qquad R^2 = 4^2 + 6^2$

$\alpha = 0.588$ (3sf) $\qquad R = \sqrt{4^2+6^2}$

$= 2\sqrt{13}$

$4\cos\theta + 6\sin\theta = 2\sqrt{13}\sin(\theta + 0.588)$

d) $2\sin 2\theta - 3(\cos 2\theta + \sin\theta - 1) = 0$

$\sin\theta(4\cos\theta + 6\sin\theta - 3) = 0$

$\sin\theta(2\sqrt{13}\sin(\theta + 0.588) - 3) = 0$

$\sin\theta = 0$ $\qquad \sin(\theta + 0.588) = \frac{3}{2\sqrt{13}}$

$\underline{\underline{\theta = 0}}$

$\theta + 0.588 = 0.4290698494,$

2.712522804

$\underline{\underline{\theta = 2.12}}$

21 a) $3\sin x + 2\cos x$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = 3$ $\quad R^2 = 2^2+3^2$

$R\sin\alpha = 2$ $\quad R = \sqrt{2^2+3^2}$

$= \sqrt{13}$

$\tan\alpha = 2/3$

$\alpha = 0.588$ (3sf)

$\cancel{R}\cdot \sqrt{13}\sin(x+0.588)$

b) $(\sqrt{13})^4 = \underline{169}$

c) $\sqrt{13}\sin(x+0.588) = 1$

$\sin(x+0.588) = \frac{1}{\sqrt{13}}$

$x+0.588 = 0.2810349015,$

$2.860557752,$

6.564220209

$\underline{x = 2.273,\ 5.976}$