CAP 5993/CAP 4993
Game Theory

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Homework

- Proof of von Neumann Theorem:
  - Base case
  - Main case: first see whether white has A winning strategy, then check whether black can ensure a win FOR ALL strategies that follow.
- Technique: can convert game to equivalent and then apply von Neuman Theorem.
- To show a statement is true, you need to give a proof.
- To show a statement is false, you need to give a counterexample.

Trembling-hand perfect equilibrium

	L	**R**
U	1, 1	2, 0
D	0, 2	2, 2

- Two pure strategy equilibria (U,L) and (D,R).
- Assume row player is playing $(1-\varepsilon, \varepsilon)$ for $0 < \varepsilon < 1$ …

- In game theory, **trembling hand perfect equilibrium** is a refinement of Nash equilibrium due to Reinhard Selten. A trembling hand perfect equilibrium is an equilibrium that takes the possibility of off-the-equilibrium play into account by assuming that the players, through a "slip of the hand" or **tremble,** may choose unintended strategies, albeit with negligible probability.

- First we define a **perturbed game**. A perturbed game is a copy of a base game, with the restriction that only totally mixed strategies are allowed to be played. A totally mixed strategy is a mixed strategy where *every* pure strategy is played with non-zero probability. This is the "trembling hands" of the players; they sometimes play a different strategy than the one they intended to play. Then we define a strategy set S (in a base game) as being trembling hand perfect if there is a sequence of perturbed games that converge to the base game in which there is a series of Nash equilibria that converge to S.

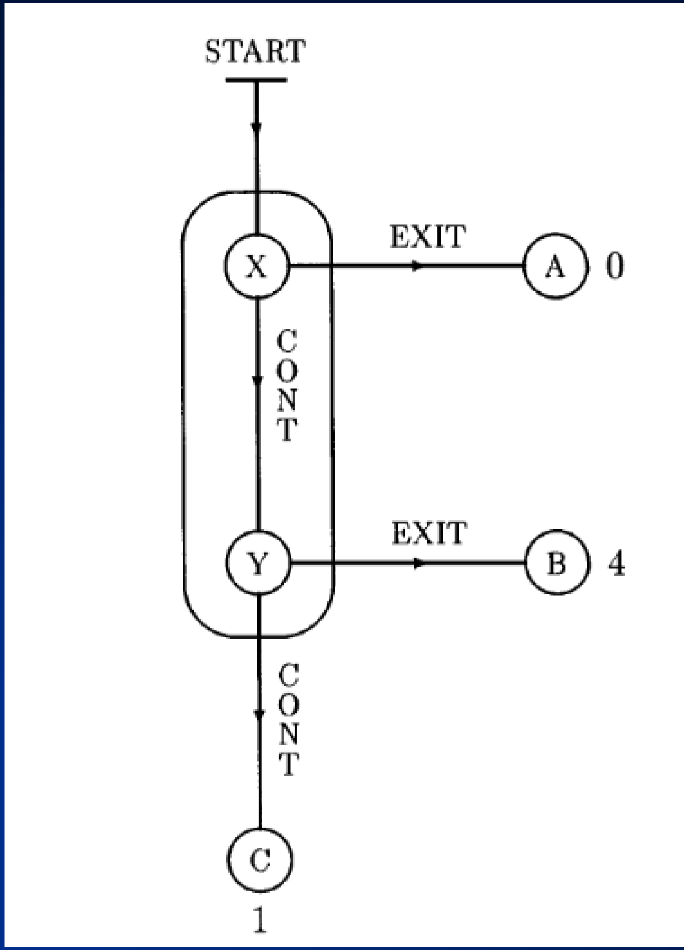
Extensive-form games

- Two ways of defining trembling hand perfect equilibrium:
  - every strategy of the extensive-form game must be played with non-zero probability. This leads to the notion of a **normal-form trembling hand perfect equilibrium**.
  - every move at every information set is taken with non-zero probability. Limits of equilibria of such perturbed games as the tremble probabilities goes to zero are called **extensive-form trembling hand perfect equilibria**.
- These two notions are incomparable.

- Theorem: Every finite strategic-form game has at least one perfect equilibrium.
- Theorem: In every perfect equilibrium, every (weakly) dominated strategy is chosen with probability zero.
- Theorem: Every equilibrium in completely mixed strategies in a strategic-form game is a perfect equilibrium.

- Theorem: Every extensive-form game has a strategic-form perfect equilibrium.
- Theorem: Every extensive-form perfect equilibrium of extensive-form game Γ is a subgame perfect equilibrium.
- Every finite extensive-form game with perfect recall has an extensive-form perfect equilibrium.
- Every finite extensive-form game with perfect recall has a subgame perfect equilibrium in behavior strategies.

Absent-minded driver



Evolutionarily stable strategies

- A mixed strategy x^* in a two-player symmetric game is an evolutionarily stable strategy (ESS) if for every mixed strategy x that differs from x^* there exists $\varepsilon_0 = \varepsilon_0(x) > 0$ such that, for all ε in $(0, \varepsilon_0)$,

$$(1-\varepsilon)u_1(x,x^*) + \varepsilon u_1(x,x) < (1-\varepsilon)u_1(x^*,x^*) + \varepsilon u_1(x^*,x)$$

- Interpret x^* as distribution of types among “normal” individuals. Consider a mutation making use of strategy x, and assume that the proportion of this mutation in the population is ε.
- In ESS, the expected payoff of the mutation is smaller than the expected payoff of a normal individual, and hence the proportion of mutations will decrease and eventually disappear over time, with the composition of the population returning to being mostly x^*. An ESS is therefore a mixed strategy of the column player that is immune to being overtaken by mutations.

What are ESS of Prisoner's dilemma?

	D	C
D	1, 1	4, 0
C	0, 4	3, 3

	Dove **y**	**Hawk** **1-y**
Dove	4, 4	2, 8
Hawk	8, 2	1, 1

- Suppose that a particular animal can exhibit one of two possible behaviors: aggressive behavior or peaceful behavior. We will describe this by saying that there are two types of animals: hawks (aggressive) and doves (peaceful). The different types of behavior are expressed when an animal invades the territory of another animal of the same species. A hawk will aggressively repel the invader. A dove, in constrast, will yield to the aggressor and be driven out of its territory. If one of the two animals is a hawk and the other a dove, the outcome of this struggle is that the hawk ends up in the territory, while the dove is driven out, exposed to predators, and other dangers. If both animals aredoves, one of them will end up leaving the territory. Suppose that each of them leaves in that situation with prob ½. If both are hawks, a fight ensues, during which both are injured, and at most one will remain in the territory and produce offspring.

- Note that the game is symmetric. A mutation is an individual in the population characterized by a particular behavior: it may be of type dove or type hawk. Mutation type x ($0 <= x <= 1$), dove prob x, hawk prob 1-x. Expected number offspring depends on its type and type of individual it encounters (prob y of dove).

- Expected payoff of mutation is $4y + 2(1-y)$ for dove, $8y + (1-y)$ for hawk, and $x(4y + 2(1-y))+(1-x)(8y+(1-y))$ if it is type x.

- Eg 80% doves ($y = 0.8$) and 20% hawks, and a new mutation is called upon, what “should” the mutation choose?

- Dove: 0.8 * 0.4 + 0.2*2 = 3.6
- Hawk: 0.8*8 + 0.2*1 = 6.6
- Mutation’s advantage to be born a hawk.
- Over the generations, number of hawks will rise and ration of doves to hawks will not be 80%/20%. So population of 80% doves:20% hawks is evolutionarily unstable.
- Similarly if 10% doves we are unstable.
- It can be shown that for 20% doves and 80% hawks, the expected number of offspring of each type will be equal.
- Note that y* = 0.2 is the symmetric equilibrium of the game.

- Theorem: If x* is an evolutionarily stable strategy in a two-player symmetric game, then (x*,x*) is a symmetric Nash equilibrium in the game.

- Theorem: A strategy x* is evolutionarily stable if and only if for each x != x* only one of the following two conditions obtains:

$$u_1(x,x^*) < u_1(x^*,x^*),$$

or

$$u_1(x,x^*) = u_1(x^*,x^*) \text{ and } u_1(x,x) < u_1(x^*,x),$$

- First condition states that if a mutation deviates from x*, it will lose in its encounters with the normal population. The second condition says that if the payoff a mutation receives from encountering a normal individual is equal to that received by a normal individual encountering a normal individual, that mutation will receive a smaller payoff when it encounters the same mutation than a normal individual would in encountering the mutation. In both cases the population of normal individuals will increase faster than the population of mutations.

Sequential equilibrium

- **Sequential equilibrium** is a refinement of Nash Equilibrium for extensive form games due to David M. Kreps and Robert Wilson. A sequential equilibrium specifies not only a strategy for each of the players but also a **belief** for each of the players. A belief gives, for each information set of the game belonging to the player, a probability distribution on the nodes in the information set. A profile of strategies and beliefs is called an **assessment** for the game. Informally speaking, an assessment is a perfect Bayesian equilibrium if its strategies are sensible given its beliefs **and** its beliefs are confirmed on the outcome path given by its strategies. The definition of sequential equilibrium further requires that there be arbitrarily small perturbations of beliefs and associated strategies with the same property.

Proper equilibrium

- **Proper equilibrium** is a refinement of Nash Equilibrium due to Roger B. Myerson. Proper equilibrium further refines Reinhard Selten's notion of a trembling hand perfect equilibrium by assuming that more costly trembles are made with significantly smaller probability than less costly ones.

- Given a normal form game and a parameter $\epsilon > 0$, a totally mixed strategy profile σ is defined to be ϵ-proper if, whenever a player has two pure strategies s and s' such that the expected payoff of playing s is smaller than the expected payoff of playing s' (that is $u(s, \sigma_{-i}) < u(s', \sigma_{-i})$), then the probability assigned to s is at most ϵ times the probability assigned to s'. A strategy profile of the game is then said to be a proper equilibrium if it is a limit point, as ϵ approaches 0, of a sequence of ϵ-proper strategy profiles.

Matching pennies with a twist

	Guess heads up	Guess Tails up	Grab penny
Hide Heads Up	-1,1	0,0	-1,1
Hide Tails Up	0,0	-1,1	-1,1

- The Nash equilibria of the game are the strategy profiles where Player 2 grabs the penny with probability 1. Any mixed strategy of Player 1 is in (Nash) equilibrium with this pure strategy of Player 2. Any such pair is even trembling hand perfect. Intuitively, since Player 1 expects Player 2 to grab the penny, he is not concerned about leaving Player 2 uncertain about whether it is heads up or tails up. However, it can be seen that the unique proper equilibrium of this game is the one where Player 1 hides the penny heads up with probability 1/2 and tails up with probability 1/2 (and Player 2 grabs the penny). This unique proper equilibrium can be motivated intuitively as follows: Player 1 fully expects Player 2 to grab the penny. However, Player 1 still prepares for the unlikely event that Player 2 does not grab the penny and instead for some reason decides to make a guess. Player 1 prepares for this event by making sure that Player 2 has no information about whether the penny is heads up or tails up, exactly as in the original Matching Pennies game.

Critiques of Nash equilibrium

- Is it too strict?
  - Does not exist in all games
  - Might rule out some more “reasonable” strategies
- Not strict enough?
  - Potentially many equilibria to select through
- Just right?

Repeated games

- In many cases, interaction between players does not end after only one encounter; players often meet each other many times, either playing the same game over and over again, or playing different games. There are many examples of situations that can be modeled as multistage interactions: a printing office buys paper from a paper manufacturer every quarter; a tennis player buys a pair of tennis shoes from a shop in his town every time his old ones wear out; baseball teams play each other several times every season.

- The very fact that the players encounter each other repeatedly gives them an opportunity to cooperate, by conditioning their actions in every stage on what happened in previous stages. A player can threaten his opponent with the threat “if you do not cooperate now, in the future I will take actions that harm you,” and he can carry out this threat, thus “punishing” his opponent. For example, the manager of a printing office can inform a paper manufacturer that if the price of the paper is not reduced by 10% in the future, he will no longer buy paper from that manufacturer.

- $\Gamma = (N, (S_i)\ i \text{ in } N, (u_i)\ i \text{ in } N)$
- Players play Γ over and over.
- Three cases:
  - Finite number of stages T, and every player wants to maximize his average payoff.
  - The game lasts an infinite number of stages, and every player wants to maximize the upper limit of his average payoffs
  - The game lasts an infinite number of stages, and each player wants to maximize the time-discounted sum of his payoffs.
- Let $M = \max_{i \text{ in } N} \max_{s \text{ in } S} |u_i(s)|$

	D	C
D	1, 1	4, 0
C	0, 4	3, 3

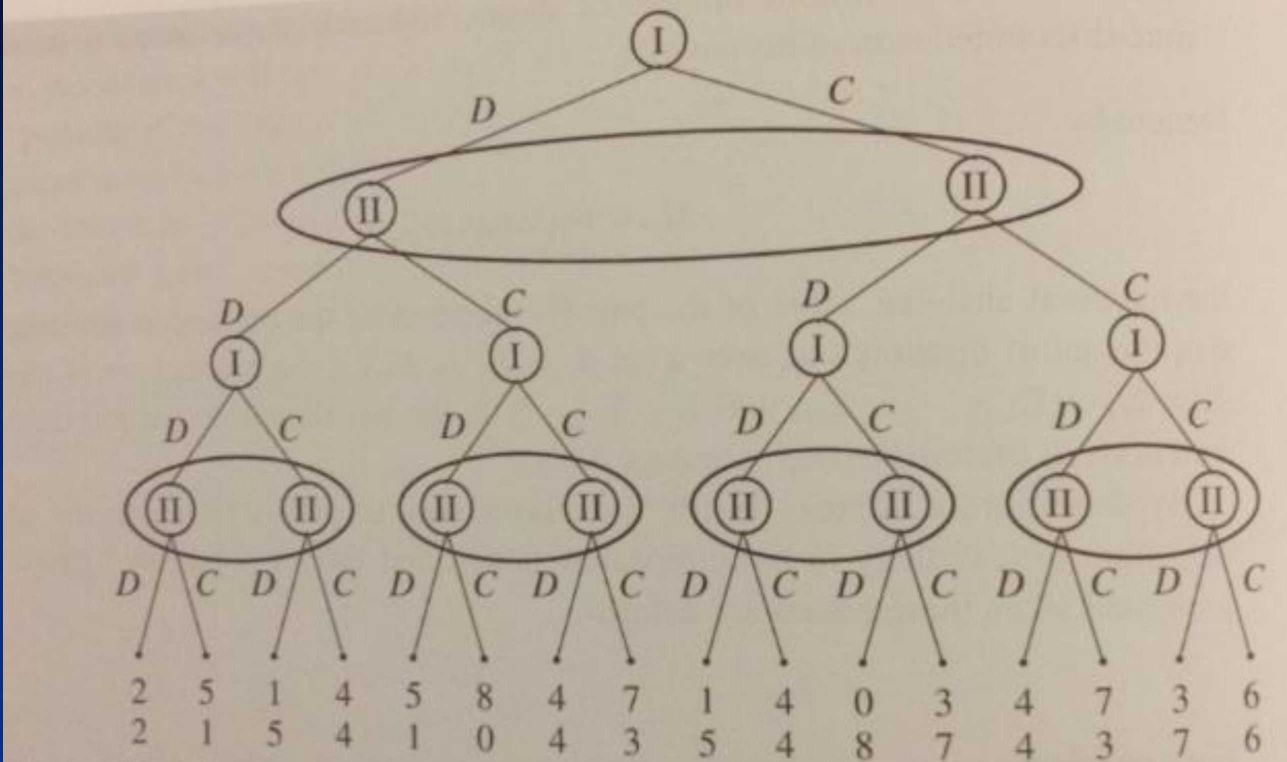


Figure 13.2 The two-stage Prisoner's Dilemma, represented as an extensive-form game

- At every equilibrium of the two-stage repeated game, the players play (D,D) in both stages.
- Proof:
  - Suppose instead there exists an equilibrium in which the players do not play (D,D) with positive probability in some stage. Let t in {1,2} be the last stage in which there is positive probability they do not play (D,D) and suppose that in this event, Player I does not play D at stage t. This means that if the game continues after stage t the players will play (D,D). We will show that this strategy cannot be an equilibrium strategy.

- Case 1: t = 1.
  - Consider the strategy of Player I at which he plays D in both stages. We will show that this strategy grants him a higher payoff. Since D strictly dominates C, Player I’s payoff rises if he switches from C to D in the first stage. And since, by assumption, after stage t the players play (D,D) (since stage t is the last stage in which they may not play (D,D)), Player I’s payoff in the second stage was supposed to be 1. By playing D in the second stage, Player I’s payoff is either 1 or 4 (depending on whether Player II plays D or C); in either case, Player I cannot lose in the second stage. The sum total of Player I’s payoffs therefore rises.

- Case 2: $t = 2$.
  - Consider the strategy of Player I at which he plays in the first stage what the original strategy tells him to play, and in the second stage he plays D. Player I’s payoff in the first stage does not change, but because D strictly dominates C, his payoff in the second stage does increase. The sum total of Player I’s payoffs therefore increases.

Next lecture

Assignment

- HW2 due today.
- HW3 out 2/21 (due 3/2).
- Midterm on 3/7 (midterm review on 3/2).
  - Will cover material from lectures and homeworks (will not cover material from the textbooks that was not covered in lectures or homeworks).
  - 3 parts: multiple choice, true/false with explanation, analytical exercises
- Reading for next class: chapter 7 from Shoham textbook