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Research Article

On Orthogonality Conditions for the Range and Kernel of Generalized Derivations

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Abstract

In the present work, authors reported characterization of orthogonality conditions of the range and the kernel of generalized derivations whose implementing operators are hyponormal in normed spaces. We have shown that if *A* is a subnormal operator on *H* with a cyclic vector and if *T* is abounded linear operator on *H* which commutes with *A* and *A** then *T* is hyponormal. Moreover, the range and the kernel of generalized derivations are orthogonal when the inducing operators are from the norm ideal.

Keywords: Orthogonality; Derivations; Range; Kernel.

Introduction

Let H be a separable infinite dimensional complex Hilbert space and let $B(H)$ denote the algebra of operators on H into itself. Given $A, B \in B(H)$, the generalised derivation is defined by[1] $\delta_{A,B} : B(H) \to B(H)$ (the elementary operator $\Delta_{A,B} : B(H) \to B(H)$) is defined by [2] $\delta_{A,B}(X) = AX - BX$, for all $X \in B(H)$. Let $d_{A,B}$ denote either $\delta_{A,B}$ or $\Delta_{A,B}$. Recall [3] that if M and N are subspaces of a Banach space V with norm $\|.\|$, M is said to be orthogonal to N if $\| m + n \| \geq \| n \|$, for all $m \in M$ and $n \in N$.

The range-kernel orthogonality of the operator $d_{A,B}$ has been considered by a number of authors in the recent past in [4], [5], [6], [7] and some of the references there in, with the first such result proved in [8]. In [9] it was proved that if $A \in B(H)$ is a normal operator and $S \in B(H)$ is in the commutant of A, then $\| \delta_{A,A}(X) + S \| \geq \| S \|$, for all $X \in B(H)$. This result has a $\Delta_{A,A}$ analogue: indeed it is known that if A and B^* satisfy a normality-like hypothesis and $d_{A,B}(S) = 0$ for some $S \in B(H)$, then $\| d_{A,B}(X) + S \| \geq \| S \|$, for all $X \in B(H)$. The reader is referred to [10] and [11] for further details.

The range-kernel orthogonality of $d_{A,B}|_{C_p}$, the restriction of $d_{A,B}$ to C_p, and more generally for the class of unitarily invariant norms, has been considered in a number of papers for instance, in [5], [8] and [9]. Given that if $S \in C_p$, for some $1 < p < \infty$, then $min\{\| d_{A,B}(X) + S_p, d_{A^*,B^*}(X) + S_p \|\} \geq \| S_p \|$ for all $X \in C_p$ if and only if $d_{A,B}(S) = 0 = d_{A^*,B^*}(S)$ [5, Theorem (iii)]. Let $\boldsymbol{A} = (A_1, A_2, \dots, A_n)$ *and* $\boldsymbol{B} = (B_1, B_2, \dots, B_n)$ be *n*-tuples of operators and define the elementary operators $\Delta_{A,B}$ and $\Delta^*_{A,B} : B(H) \to B(H)$ (of length $n+1$) by $\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$ and $\Delta^*_{A,B}(X) = \sum_{i=1}^{n} A^*_i X B^*_i - X$. The range-kernel orthogonality of $\Delta_{A,B}|_{C_p}$, $1 \leq p < \infty$, has recently been considered in [10], where it is shown that if $\sum_{i=1}^{n} A^*_i A_i$, $\sum_{i=1}^{n} A_i A^*_i$, $\sum_{i=1}^{n} B^*_i B_i$ and $\sum_{i=1}^{n} B_i B^*_i$ are all less than 1, and if $\Delta_{A,B}(S) = 0 = \Delta^*_{A,B}(S)$, then $min\{\| \Delta_{A,B}(X) + S \|_p, \| \Delta^*_{A,B}(X) + S \|_p\} \geq \| S \|_p$, for all $X \in C_p$. Here, the *n*-tuples A, B consist of mutually commuting normal operators. This paper characterizes the conditions of orthogonality of the range and the kernel of generalized derivations. The implementing operators, in this case, are hyponormal operators.

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Preliminaries

In this section, we give preliminary concepts which are useful in this paper. We begin with the following definition.

Definition 2.1. Let $T \in N_H(H)$, the class of all norm attainable hyponormal operators then the kernel of the operator $T, Ker(T) = \{X \in N_H(H): T(X) = 0, for\ all\ X \in N_H(H)\}$ and the range of the operator $T, Ran(T) = \{X \in N_H(H): X = T(X), for\ all\ X \in N_H(H)\}$

Definition 2.2. The vector x is a smooth point of the sphere $S(0, \| x \|)$ if there exists a unique support functional $F_x \in X^*$, such that $F_x(x) = \| x \|$ and $\| F_x \| = 1$.

Definition 2.3. Let $(A_1, A_2, \ldots, A_n)$ and $(B_1, B_2, \ldots, B_n)$ be the *n*-tuples of bounded Hilbert space operators. The mapping $X \to \sum_{j=1}^{n} A_j B_j$ from $B(H)$ to $B(H)$ is called the elementary operator or elementary mapping.

Definition 2.4. An element $S \in B(H)$ is a commutator if there exist $X, T \in B(H)$ such that $S = XT - TX$; is positive if $\| S \| \geq 0$; hyponormal if $SS^* = S^*S$; and self-adjoint if $S = S^*$.

Definition 2.5. For $A, B \in B(H)$, let $\delta_{A,B}$ denote the operator on $B(H)$ defined by $\delta_{A,B}(X) = AX - XB$ is called the generalized derivation.

Definition 2.6. Let $A, B \in B(H)$. We say that the pair A, B satisfies $(FP)_{B(H)}$ the Fuglede-Putnam's property, if $AC = CB$ where $C \in B(H)$ implies $A^*C = CB^*$.

Results and discussion

In this section, we give results on the range-kernel orthogonally of the generalized derivations implemented by hyponormal operators in normed spaces. We consider the class of all norm attainable hyponormal operators on the Hilbert space H, which we denote by $N_H(H)$. We begin with the following auxiliary lemma;

Lemma 3.1. *Let* $A, B \in N_H(H)$. Then the following statements are equivalent:

(i). The pair (A, B) has the property $(FP)_{N_H(H)}$, $1 \leq p < \infty$.

(ii). If $AT = TB$ where $T \in B(H)$, then $\overline{R(T)}$ reduces A, $Ker(T)^{\perp}$ reduces B, and $A|_{\overline{R(T)}}$ and $B|_{KerT^{\perp}}$ are hyponormal operators.

Proof. $(i) \Rightarrow (ii)$. Since $AT = TB$ and (A, B) satisfies $(FP)_{N_H}(H)$, $A^*T = TB^*$, $\overline{R(T)}$ and $Ker(T)^{\perp}$ are reducing subspaces for A and B respectively. Since $A(AT) = (AT)B$ implies that $A^*(AT) = (AT)B^*$, a result from the $(FP)_{N_H}(H)$. The identity $A^*T = TB^*$ also implies that $A^*AT = AA^*T$. Thus, $A|_{\overline{R(T)}} = B|_{Ker(T)^{\perp}}$ is hyponormal.

$(ii) \Rightarrow (i)$. Let $T \in N_H(H)$ be such that $AT = TB$. Taking the two decompositions of H, such that $H_1 = H = \overline{R(T)} \oplus R(T)^{\perp}$ and $H_2 = H = Ker(T)^{\perp} \oplus Ker(T)$, then we can write A and B on H_1 and H_2 respectively as $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$, where A_1, B_1 are hyponormal operators. Also, let T and X on H_1 into H_2 have operator representation $T \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$. It follows from $AT = TB$ that $A_1T_1 = T_1B_1$. Since A_1 and B_1 are hyponormal operators, from the Fuglede-Putnam's hypothesis, we have that $A^*{}_1T_1 = T_1B^*{}_1$ an implication that $A^*T = TB^*$.

Theorem 3.2. Let $A, B \in N_H(H)$ satisfy $(FP)_{N_H(H)}$. Then we have that $\| T + AX - XB \| \geq \| T \|$, for every $T \in Ker(\delta_{A,B})$ and for all $X \in B(H)$.

Proof. If the pair (A, B) satisfy $(FP)_{N_H(H)}$ property, then $\overline{Ran(T)}$ reduces A, $Ker(T)^{\perp}$ reduces B, and $A|_{\overline{Ran(T)}}$ and $B|_{Ker^{\perp}T}$ are hyponormal operators. Letting $T_0 : Ker^{\perp}(T) \to \overline{Ran(T)}$ be the quasiaffinity defined by setting $T_0x = Tx$, for each $x \in Ker^{\perp}(T)$, we have that $\delta_{A_1,B_1}(T_0) = \delta_{A^*{}_1,B^*{}_1}(T_0) = 0$. Let $A = A_1 \oplus A_2$, with respect to the decomposition $H = Ker(T)^{\perp} \oplus Ker(T)$ and $X : \overline{R(T)} \oplus R(T)^{\perp} \to Ker(T)^{\perp} + Ker(T)$ have the operator representation $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$. Then we have

$\| T - (AX - XB) \| = \| \begin{pmatrix} T_1 - (A_1X_1 - X_1B_1) & * \\ * & * \end{pmatrix} \|$.

Then $\| T - (AX - XB) \| \geq \| T_1 - (A_1X_1 - X_1B_1) \|$. Since A_1 and B_1 are hyponormal operators, it results that $\| T_1 - (A_1X_1 - X_1B_1) \| \geq \| T_1 \| = \| T \|$ which completes the proof.

Theorem 3.3. Let $A, B \in N_H(H)$. If B is invertible hyponormal operator and $\| A \| . \| B^{-1} \| \leq 1$, then $\| \delta_{A,B}(X) + T \| \geq \| T \|$, for all $X \in N_H(H)$ and for all $T \in Ker(\delta_{A,B})$.

Proof. Let $T \in N_H(H)$ such that $AT = TB$. This also implies that $ATB^{-1} = T$. Since $\| A \| . \| B^{-1} \| \leq 1$, it follows from $Lemma$ 3.1 that, $\| AZB^{-1} - Z + T \| \geq \| T \| \ldots.. (*)$, for all $Z \in N_H(H)$. Setting $X = ZB^{-1}$, then Inequality $(*)$ becomes $\| AX - XB + T \| \geq \| T \|$. Hence, $\| \delta_{A,B}(X) + T \| \geq \| T \|$, for all $T \in Ker(\delta_{A,B})$ and for all $X \in B(H)$ which completes the proof.

Theorem 3.4. Let $A, B \in N_H(H)$. If either;

(i). A is an isometric hyponormal and B is a contraction or,

(ii). A is a contraction and B is co-isometric hyponormal, then $\| \delta_{A,B}(X) + T \| \geq \| T \|$, for all $X \in B(H)$ and for all $T \in Ker(\delta_{A,B})$.

Proof. (1). Given that $T \in Ker(\delta_{A,B})$, we have that $\delta_{A,B}(T) = 0$ which implies that $T = A^*TB$ which further implies that $A^*T = A^*(A^*T)B$. Moreover, we have that $\| \delta_{A,B}(X) + T \| \geq \| A^*(\delta_{A,B}(X) + T) \| = \| \Delta_{A^*,B}(X) - A^*T \|$. Since A is an isometric hyponormal and B is a contraction, it follows from $Lemma$ 3.1 that $\| \delta_{A,B}(X) + T \| \geq \| \Delta_{A^*,B}(X) - A^*T \| \geq \| A^*T \| \geq \| A^*TB \| = \| T \|$. Then, $\| \delta_{A,B}(X)T \| \geq \| T \|$, for all $X \in B(H)$.

(2). Let $T \in Ker(\delta_{A,B})$ and $X \in B(H)$. By taking adjoints, we have that $\| \delta_{A,B}(X) + T \| = \| \delta^*_{B,A^*}(X^*) - T^* \|$. Since B^* is isometric hyponormal and A^* is a contraction, the result follows from the first part of the proof.

Lemma 3.5. If A is a subnormal operator on H with a cyclic vector and if T is abounded linear operator on H which commutes with A and A^*, then T is hyponormal.

Proof. By [12], we have that T and T^* are subnormal. Since every subnormal operator T on H is hyponormal, i.e, $(\| Sx \| \geq \| S^*x \|)$, for all $x \in H$, then T is hyponormal.

Theorem 3.6. Let $(J, \|.\|_J)$ be a norm ideal and $A \in N_H(H)$. Suppose that $f(A)$ is a hyponormal operator where f is a nonconstant analytic function on an operator set containing the $\sigma(A)$. Then $\| \delta_A(X) + T \|_J \geq \| T \|_J$, for all $X \in J$ and for all $T \in \{A\}' \cap J$.

Proof. Let $T \in J$ be such that $AT = TA$, then we have that $f(A)T = Tf(A)$ and $Af(A) = f(A)A$. Since $f(A)$ is hyponormal operator, it follows from $Lemma$ 3.5 that T and A are subnormal. Therefore, every compact subnormal operator is hyponormal and hence T is hyponormal. Consequently, $AT = TA$ implies that $AT^* = T^*A$ and hence we obtain $Ran(T)$ and $Ker^{\perp}(T)$ reduce A and $A_0 = A|_{\overline{R(T)}}$ and $B_0 = B|_{Ker^{\perp}(T)}$ are hyponormal operators. Let $A = A_0 \oplus A_1$ with respect to the decomposition $H_0 = H = \overline{Ran(T)} \oplus R(T)^{\perp}$ and let $B = B_0 \oplus B_1$ with respect to the decomposition $H_1 = H = Ker^{\perp}(T) \oplus Ker(T)$. Define a quasi-affinity $T_0 : Ker^{\perp}(T) \to \overline{Ran(T)}$ by setting $T_0x = Tx$, for every $x \in Ker^{\perp}(T)$. Then it results that $\delta_{A_0,B_0}(T_0) = \delta_{A^*_0,B^*_0}(T_0) = 0$. Also, let T and X on H_1 into H_0 have operator representation $T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix}, X = \begin{pmatrix} X_0 & X_1 \\ X_2 & X_3 \end{pmatrix}$ and as a consequence we have that $\| \delta_A(X) + T \|_J = \| \begin{pmatrix} \delta_{A_0,B_0}(X_0) + T_0 & * \\ * & * \end{pmatrix} \| \geq \| \delta_{A_0,B_0}(X_0) + T \|_J$. Since A_0 and B_0 are hyponormal operators, we obtain from [4] that;

$$\| \delta_A(X) + T \|_J \geq \| \delta_{A_0,B_0}(X) + T_0 \|_J \geq \| T_0 \|_J = \| T \|_J$$

Corollary. 3.7. Let $A, B \in N_H(H)$. If the pair (A, B) possesses the $PF(\Delta, J)$ property, then $\| \Delta_{A,B}(X) + T \|_J \geq \| T \|_J$, for all $X \in J$ and for all $T \in Ker(\Delta_{AB} \cap J)$.

Proof. Let $T \in J$ such that $ATB = T$. Since the pair (A, B) possesses the $PF(\Delta, J)$ property, $\overline{Ran(T)}$ reduces A and $Ker^{\perp}(T)$ reduces B and $A_0 = A|_{R(T)}, B_00 = B|_{Ker^{\perp}}(T)$ are hyponormal operators. Let $T_0 = Ker^{\perp}(T) \to \overline{Ran(T)}$ be the quasiaffinity defined by setting $T_0x = Tx$, for each $x \in Ker^{\perp}(T)$. Then we have that $\Delta_{A_0,B_0}(T_0) = 0 = \Delta_{A^*_0,B^*_0}(T0)$. Let $A_0 \oplus A_1$ with respect to the decomposition $H_0 = H = \overline{Ran(T)} \oplus \overline{Ran(T)}^{\perp}$ and $B = B_0 \oplus B_1$ with respect to the decomposition $H_1 = H = Ker^{\perp}(T) \oplus Ker(T)$. Let X be on H_1 into H_2 have the operator representation

$X = \begin{pmatrix} X_0 & X_1 \\ X_2 & X_3 \end{pmatrix}$. Then

$$\| \Delta_{A,B}(X) + T \|_J = \| \begin{pmatrix} \Delta_{A_0,B_0}(X_0) + T_0 & * \\ * & * \end{pmatrix} \|_J$$

Since the diagonal part of a block matrix always has a smaller norm than that of the whole matrix, we have that

$$\| \Delta_{A,B}(X) + T \|_J = \| \begin{pmatrix} \Delta_{A_0,B_0}(X_0) + T_0 & * \\ * & * \end{pmatrix} \|_J \geq \| \Delta_{A_0,B_0}(X_0) + T_0 \|_J.$$

Conclusion

In this paper, authors have characterized the orthogonality conditions of the range and kernel of generalized derivations implemented by hyponormal operators in normed spaces. We based our study on the class of all norm attainable hyponormal operators $N_H(H)$.

Conflict of interest

The authors declare no conflict of interest.

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