So consider $u_x - u_y = 0$

and the change of variables

$r = x+y \quad s = x-y$

so $u_x = u_r r_x + u_s s_x \qquad u_y = u_r r_y + u_s s_y$

$= u_r + u_s \qquad\qquad = u_r - u_s$

so $u_x - u_y = u_r + u_s - (u_r - u_s)$

$= 2u_s = 0$

so $u_s = 0 \Rightarrow u = f(r)$

$= f(x+y)$

Sol^n $u = f(x+y)$

New change of variables

$r = x+y\ ,\ s = x^2 - y^2$ $\quad$ Note x^2-y^2
$= (x-y)(x+y)$

$u_x = u_r r_x + u_s s_x$

$= u_r + 2x\, u_s\ ,$

$u_y = u_r r_y + u_s s_y$

$= u_r - 2y u_s$

$u_x - u_y = 0 \Rightarrow \cancel{u_r} + 2x u_s - \cancel{u_r} + 2y u_s = 0$

$\Rightarrow 2(x+y) u_s = 0$

$\Rightarrow u_s = 0 \Rightarrow u = f(r)$

$= f(x+y)$ same.

Cofr 3 $\quad r = x-y,\ s = x^2-y^2$

$u_x = u_r + 2x u_s$

$u_y = -u_r - 2y u_s$

so $u_x - u_y = u_r + 2x u_s - (-u_r - 2y u_s)$

$= u_r + 2x u_s + u_r + 2y u_s$

$= \cancel{2} u_r + \cancel{2}(x+y) u_s = 0$

Now put everything in terms of r, s'

$u_r + (x+y) u_s = 0$ $\qquad r = x-y \quad s = x^2 - y^2$
$= (x-y)(x+y)$

$\Rightarrow u_r + \frac{s}{r} u_s = 0$ $\qquad \Rightarrow x+y = \frac{s}{r}$

or $r u_r + s u_s = 0$

A harder PDE

So (1) $r = x+y \quad s = x-y$ ✓

(2) $r = x+y \quad s = x^2 - y^2$ ✓

3 $r = x-y \quad s = x^2 - y^2$ X

so if $r = x+y$, $s = s(x,y)$ ∃ should write

$$u_x = u_r + s_x u_s$$

$$u_y = u_r + s_y u_s$$

$$u_x - u_y = 0 \Rightarrow (s_x - s_y) u_s = 0$$

(note $s_x - s_y \neq 0$) more later

$$u_s = 0 \Rightarrow u = f(r)$$

$$= f(x+y)$$

what about if we don't know r

$$u_x = u_r r_x + u_s s_x$$

$$u_y = u_r r_y + u_s s_y$$

$$u_x - u_y = 0 \qquad (r_x - r_y) u_r + (s_x - s_y) u_s = 0$$

choose $r_x - r_y = 0 \Rightarrow (s_x - s_y) u_s = 0$

$u_s = 0$ as before

need to solve

so to solve our PDE

$$u_x - u_y = 0$$

is to solve $r_x - r_y = 0$???

Instead we will walk backwards

$$u_x - u_y = 0 \Rightarrow u_s = 0$$

chain Rule

$$u_s = u_x x_s + u_y y_s = 0$$

pick $x_s = 1$, $y_s = -1$ so $u_s = u_x - u_y = 0$

so $x_s = 1$, $y_s = -1$, $u_s = 0$ ← we can solve these

$$\left.\begin{aligned} x &= s + a(r) \\ y &= -s + b(r) \\ u &= c(r) \end{aligned}\right\} \text{ sol}^n \text{ is parametric}$$

$$x+y = a(r) + b(r) = A(r)$$

$$r = A^{-1}(x+y)$$

$$u = c(r) = c(A^{-1}(x+y)) \qquad cA^{-1}(*) = f(*)$$

$$\boxed{u = f(x+y)}$$ sdn as before

Ex 2 $\quad 2u_x + u_y = 4$

start $\quad u_s = u_x x_s + u_y y_s$

choose $\quad x_s = 2,\ y_s = 1$ so $\quad u_s = 4$

$$\begin{aligned} x &= 2s + a(r) \\ y &= s + b(r) \\ u &= 4s + c(r) \end{aligned}$$

$$x - 2y = a(r) - 2b(r) = A(r)$$

$$u = 4(y - b(r)) + c(r)$$

$$= 4y + c(r) - 4b(r)$$

$$u = 4y + d(r)$$

so $u = 4y + d\left(A^{-1}(x-2y)\right)$

$$\boxed{u = 4y + f(x-2y)}$$

check

$u_x = f'(x-2y)$

$u_y = 4 + f'(x-2y)(-2)$

L.S. $2u_x + u_y$

$= \cancel{2f'(x-2y)} + 4 - \cancel{2f'(x-2y)}$

$= 4$

$=$ R.S.