Math 2471 Calculus III – Sample Test 2 Solutions

1. Classify the critical points for the following

$$\begin{aligned} (i) \quad z &= x^3 + y^3 - 3x - 12y + 20 \\ (ii) \quad z &= 3xy - x^2y - xy^2 \end{aligned}$$

Soln
(i) Calculating the partial derivatives gives

$$z_x = 3x^2 - 3 = 3(x-1)(x+1), \quad z_y = 3y^2 - 12 = 3(y-2)(y+2)$$

and the critical points are when $z_x = 0$, $z_y = 0$ giving $x = \pm 1$ and $y = \pm 2$ leading to the critical points $(-1,-2)$, $(-1,2)$, $(1,-2)$, and $(1,2)$. To determine the nature of the critical points we use the second derivative test. So

$$z_{xx} = 6x, \quad z_{xy} = 0, \quad z_{yy} = 6y$$

giving

$$\Delta = z_{xx}z_{yy} - z_{xy}^2 = 36xy$$

Now, we consider each critical point separately.

$$\begin{array}{rll} (-1,-2) & \Delta = 72 > 0, \ z_{xx} < 0 & \text{max} \\ (-1,2) & \Delta = -72 < 0 & \text{saddle} \\ (1,-2) & \Delta = -72 < 0 & \text{saddle} \\ (1,2) & \Delta = 72 > 0, \ z_{xx} > 0 & \text{min} \end{array}$$

(ii) Calculating the partial derivatives gives

$$z_x = 3y - 2xy - y^2 = y(3 - 2x - y), \quad z_y = 3x - x^2 - 2xy = x(3 - x - 2y)$$

and the critical points are when $z_x = 0$, $z_y = 0$ giving the critical points $(0,0)$, $(0,3)$, $(3,0)$, and $(1,1)$. To determine the nature of the critical points we use the second derivative test. So

$$z_{xx} = -2y, \quad z_{xy} = 3 - 2x - 2y, \quad z_{yy} = -2x$$

giving

$$\Delta = z_{xx}z_{yy} - z_{xy}^2 = 4xy - (3 - 2x - 2y)^2$$

Now, we consider each critical point separately.

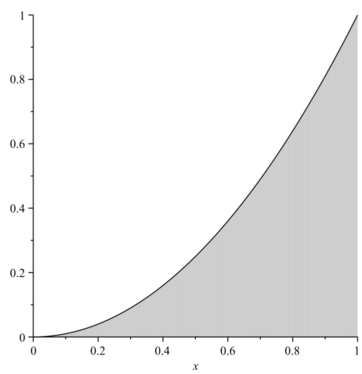
$$\begin{array}{lll} (0,0) & \Delta = -9 < 0 & \text{saddle} \\ (0,3) & \Delta = -9 < 0 & \text{saddle} \\ (3,0) & \Delta = -9 > 0 & \text{saddle} \\ (1,1) & \Delta = 3 > 0, \ z_{xx} < 0 & \text{max} \end{array}$$

2. Reverse the order of integration and integrate showing your steps.

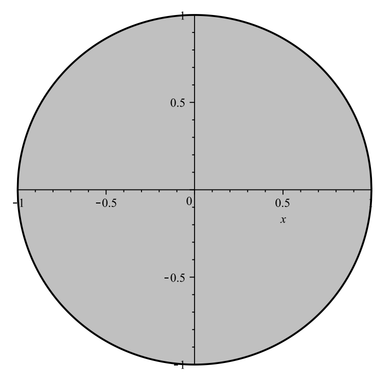
$$\int_0^1 \int_{\sqrt{y}}^1 \frac{3\,dx\,dy}{1+x^3}$$

Soln. From the region of integration (see figure 1 below) we have

$$\int_0^1 \int_0^{x^2} \frac{3\,dy\,dx}{1+x^3} = \int_0^1 \left.\frac{3y}{1+x^3}\right|_0^{x^2} dx = \int_0^1 \frac{3x^2}{1+x^3}\,dx = \ln(1+x^3)\Big|_0^1 = \ln 2$$



(a) Figure 1



(b) Figure 2

3. Find the volume bound by the paraboloid $z = 2 - x^2 - y^2$ and the cone $z = \sqrt{x^2+y^2}$

Soln. From the two surfaces we see they intersect when $z = 2 - z^2$ or $(z+2)(z-1) = 0$ giving $z = 1$ as $z = -2$ is inadmissible. The volume is then obtained from the integral

$$\iint_R \left(2 - x^2 - y^2 - \sqrt{x^2+y^2}\right) dA$$

As the region of integration is a circle of radius 1 (see figure 2 above), we switch to polar coordinates giving

$$\int_0^{2\pi} \int_0^1 \left(2 - r^2 - r\right) r dr d\theta = \frac{5\pi}{6}$$

4. Find the limits of integration of the triple integral

$$\iiint_V f(x,y,z)\,dV$$

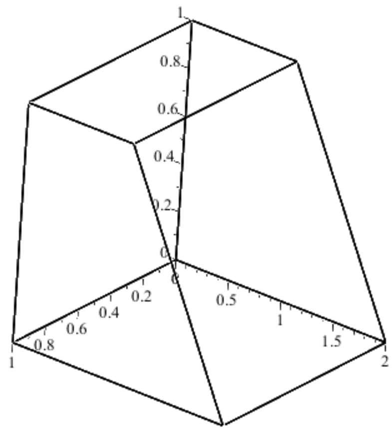
where the volume is bound by

(i) $x = 0, x = 1, y = 0, z = 0, z = 1,$ and $z = 2 - y$.

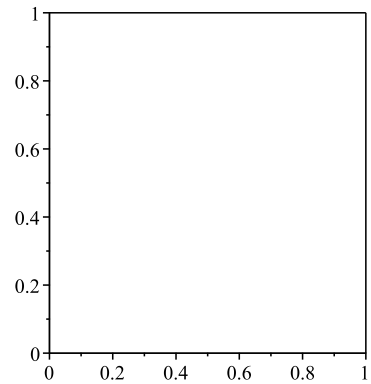
(ii) $x = 0, z = 0, z = 1 - y^2$, and $z = 2 - x$.

Soln. (i) The integral is best set up with the surface to surface going in the y direction. The outer two integrals is then over the region of the square (see figure 4).

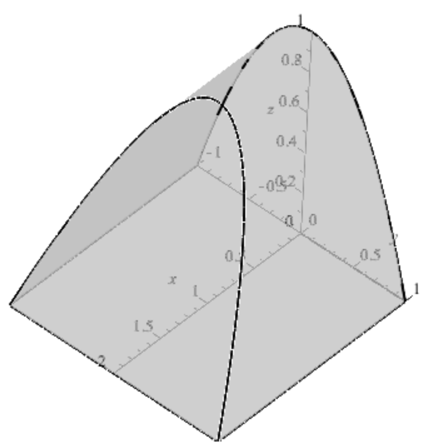
$$\int_0^1 \int_0^1 \int_0^{2-z} f(x, y, z)\, dy\, dz\, dx$$



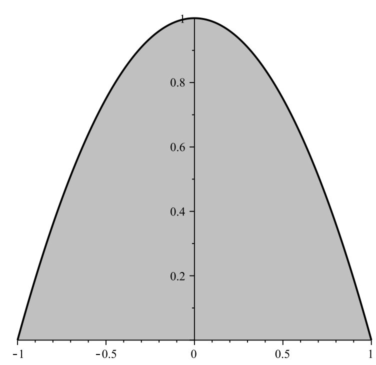
(a) figure 3



(b) figure 4



(a) figure 5



(b) figure 6

Soln. (ii) The integral is best set up with the surface to surface going in the x direction. The outer two integrals is then over the region of the parabola (see figure 6).

$$\int_{-1}^1 \int_0^{1-y^2} \int_0^{2-z} f(x, y, z)\, dx\, dz\, dy$$

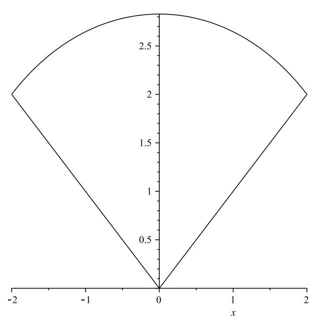
5. Set of the triple integral $\iiint\limits_V z\,dV$ in both cylindrical and spherical coordinates for the volume inside both the hemisphere $x^2 + y^2 + z^2 = 8$ and the cone $z^2 = x^2 + y^2$

Soln - Cylindrical Eliminating z between the equations gives $x^2 + y^2 = 4$. This is the region of integration (see figure (b))

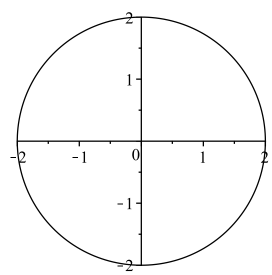
$$\int_0^{2\pi} \int_0^2 \int_r^{\sqrt{8-r^2}} z\, r\, dz\, dr\, d\theta$$

Soln - Spherical From the picture (figure (a)) we see that $\phi = 0 \to \pi/4$. Further, $\rho = 0 \to 2\sqrt{2}$ and $\theta = 0 \to 2\pi$ so

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\sqrt{2}} \rho \cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$



(a) side view.



(b) top view.