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LONGITUDINAL AND TORSIONAL OSCILLATIONS OF A ROD IN A THIRD-GRADE FLUID

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Abstract—The flow of a third-grade fluid due to the torsional and longitudinal oscillations of an infinite circular rod is discussed. The non-linear coupled partial differential equations resulting from the momentum equations are solved numerically and the result is compared with that of a regular perturbation solution. The velocity field, the axial shear force, and the torque acting on the rod are computed. The stress power is found to be negative at particular times, but that expended in a cycle turns out to be positive.

1. INTRODUCTION

Truesdell [1] suggested that, if the terms of the Clausius–Duhem inequality regarding the motion of a fluid are taken as independent, the stress power is positive. Although this is true for a purely viscous media, it does not necessarily hold for all materials. In fact, Rajagopal [2] showed that, for a dynamically possible motion taking place in a dissipative material (in this case a second-order fluid), the stress power may be negative locally in space at some instants of time. Rivlin [3] conjectured, however, that the stress power should be positive over a cycle. Huilgol [4] found a steady flow of second-grade fluid that is compatible with all the equations of rational thermodynamics and for which the stress power is negative throughout the whole domain. Consequently, deformation may generally occur with absorption or release of energy or both.

The task of this work is to determine the behavior of the stress power, the velocity field, and the dynamical boundary layer when an infinite circular rod performs longitudinal and torsional oscillations about its axis of symmetry in a third-grade fluid.

The first author to address the problem of rotational oscillations of a rod immersed in a fluid was Stokes [5], in the case of the classical linear viscous fluid. Casarella and Laura [6] considered also the longitudinal oscillations of the rod and computed the drag. Rajagopal [7] extended the problem to a second-grade fluid and found an exact solution in terms of the modified Bessel functions but did not explicitly compute the drag.

Although the second-grade fluid model is able to predict the normal stress differences which are characteristic of non-Newtonian fluids, it does not take into account the shear thinning and thickening phenomena that many show. The third-grade fluid model represents a further, although inconclusive, attempt toward a comprehensive description of the properties of viscoelastic fluids.

2. PRELIMINARIES

An incompressible simple fluid is defined as a material whose state of present stress is determined by the history of the deformation gradient without a preferred reference configuration [8]. Its constitutive equation can be written in the form of a functional

$$\mathbf{T}(t) = -p\mathbf{I} + \mathop{\mathfrak{F}}_{s=0}^{\infty}(\mathbf{F}_t^t(s)), \tag{2.1}$$

where $p\mathbf{I}$ is the undetermined part of the stress tensor and $\mathbf{F}$ is the deformation gradient.

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Coleman and Noll [9] defined the incompressible fluid of differential type of grade n as the simple fluid obeying the constitutive equation

$$\mathbf{T} = -p\mathbf{I} + \sum_{j=1}^{n} \mathbf{S}_j \tag{2.2}$$

obtained by asymptotic expansion of the functional in (2.1) through a retardation parameter α. If $n = 3$ the first three tensors $\mathbf{S}_j$ are given by

$$\mathbf{S}_1 = \mu\mathbf{A}_1,$$
$$\mathbf{S}_2 = \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$
$$\mathbf{S}_3 = \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\,\mathbf{A}_1^2)\mathbf{A}_1.$$

3. GOVERNING EQUATIONS

We will require that the Clausius–Duhem inequality is met, that is, the specific Helmholtz free energy is a minimum at the equilibrium for the system. This assumption implies that [10]

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \beta_1 = \beta_2 = 0,$$
$$\beta_3 \geq 0, \quad -\sqrt{24\mu\beta_3} \leq \alpha_1 + \alpha_2 \leq \sqrt{24\mu\beta_3}.$$

Hence, the stress tensor for an incompressible homogeneous fluid of third-grade simplifies to

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(\mathrm{tr}\,\mathbf{A}_1^2)\mathbf{A}_1, \tag{3.1}$$

where $p\mathbf{I}$ is the indeterminate part of the stress tensor due to the constraint of incompressibility, μ the viscosity, α_1, α_2 the normal stress moduli, and β_3 the higher-order viscosity.

We are going to assume that the temperature and other variables do not affect the rheological properties of the fluid, which are therefore given as constant throughout the work. The kinematic tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined through [11]

$$\mathbf{A}_1 = (\mathrm{grad}\,\mathbf{v}) + (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}}, \tag{3.2a}$$

$$\mathbf{A}_2 = \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1(\mathrm{grad}\,\mathbf{v}) + (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{A}_1, \tag{3.2b}$$

where $\mathbf{v}$ is the velocity, grad the gradient operator and $\mathrm{d}/\mathrm{d}t$ the material time derivative.

On substituting (3.1) into the momentum equation

$$\mathrm{div}\,\mathbf{T} + \rho\mathbf{b} = \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{3.3}$$

and neglecting the body forces, we have

$$\mu\,\mathrm{div}\,\mathbf{A}_1 + \alpha_1\,\mathrm{div}\,\mathbf{A}_2 + \alpha_2\,\mathrm{div}\,\mathbf{A}_1^2 + \beta_3\mathrm{div}\,[(\mathrm{tr}\,\mathbf{A}_1^2)\mathbf{A}_1] = \rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathrm{grad}\,\mathbf{p}. \tag{3.4}$$

As the rod is assumed to be infinitely long and for axisymmetric solutions, we assume a velocity field of the form

$$\mathbf{v} = v(r, t)\mathbf{e}_\theta + w(r, t)\mathbf{e}_z, \tag{3.5}$$

where $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are the unit vectors along the r, θ and z directions, respectively, and the pressure field as

$$p = p(r, t). \tag{3.6}$$

Substituting (3.5) into (3.2a) and (3.2b) we obtain

$$\mathbf{A}_1 = \begin{bmatrix} 0 & \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) & \left(\frac{\partial w}{\partial r}\right) \\ \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) & 0 & 0 \\ \left(\frac{\partial w}{\partial r}\right) & 0 & 0 \end{bmatrix}, \tag{3.7a}$$

$$\mathbf{A}_2=\begin{bmatrix} 2\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2+2\left(\frac{\partial w}{\partial r}\right)^2 & \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) & \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial r}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) & 0 & 0 \\ \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial r}\right) & 0 & 0 \end{bmatrix}. \quad (3.7b)$$

After algebraic manipulations,

$$\mathbf{A}_1^2=\begin{bmatrix} \left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2+\left(\frac{\partial w}{\partial r}\right)^2 & 0 & 0 \\ 0 & \left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2 & \left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\left(\frac{\partial w}{\partial r}\right) \\ 0 & \left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\left(\frac{\partial w}{\partial r}\right) & \left(\frac{\partial w}{\partial r}\right)^2 \end{bmatrix}. \quad (3.7c)$$

Using the above expressions in (3.1), equation (3.3) yields

$$\alpha_1\left\{4\frac{\partial v}{\partial r}\frac{\partial^2 v}{\partial r^2}-\frac{2}{r}\left(\frac{\partial v}{\partial r}\right)^2-\frac{4v}{r}\frac{\partial^2 v}{\partial r^2}-\frac{2v^2}{r^3}+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2\right.$$
$$\left.+4\left(\frac{\partial w}{\partial r}\right)\left(\frac{\partial^2 w}{\partial r^2}\right)+\frac{4v}{r^2}\frac{\partial v}{\partial r}\right\}+2\alpha_2\left\{\frac{\partial v}{\partial r}\frac{\partial^2 v}{\partial r^2}-\frac{1}{r}\left(\frac{\partial v}{\partial r}\right)^2+\frac{2v}{r^2}\frac{\partial v}{\partial r}\right.$$
$$\left.-\frac{v}{r}\frac{\partial^2 v}{\partial r^2}-\frac{v^2}{r^3}+\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+\frac{1}{2r}\left(\frac{\partial w}{\partial r}\right)^2\right\}+\rho\frac{v^2}{r}=\frac{\partial p}{\partial r}=0, \quad (3.8a)$$

$$\mu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)+2\beta_3\left\{\frac{2}{r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^3+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\right.$$
$$+3\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2\left(\frac{\partial^2 v}{\partial r^2}-\frac{1}{r}\frac{\partial v}{\partial r}+\frac{v}{r^2}\right)+2\left(\frac{\partial w}{\partial r}\right)\left(\frac{\partial^2 w}{\partial r^2}\right)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)$$
$$\left.+\left(\frac{\partial^2 w}{\partial r^2}\right)\left(\frac{\partial^2 v}{\partial r^2}-\frac{1}{r}\frac{\partial v}{\partial r}+\frac{v}{r^2}\right)\right\}-\rho\frac{\partial v}{\partial t}=\frac{1}{r}\frac{\partial p}{\partial \theta}=0, \quad (3.8b)$$

$$\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+2\beta_3\left\{\frac{1}{r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2\left(\frac{\partial w}{\partial r}\right)\right.$$
$$+\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^3+2\left(\frac{\partial w}{\partial r}\right)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\left(\frac{\partial^2 v}{\partial r^2}-\frac{1}{r}\frac{\partial v}{\partial r}+\frac{v}{r^2}\right)$$
$$\left.+\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2\left(\frac{\partial^2 w}{\partial r^2}\right)+3\left(\frac{\partial w}{\partial r}\right)^2\left(\frac{\partial^2 w}{\partial r^2}\right)\right\}-\rho\frac{\partial w}{\partial t}=\frac{\partial p}{\partial z}=0. \quad (3.8c)$$

The velocity of the surface of the rod is given by

$$\mathbf{v}(r_0,t)=V_1\cos(\omega_1 t)\mathbf{e}_\theta+V_2\cos(\omega_2 t)\mathbf{e}_z, \quad (3.9)$$

where V_1, V_2 are constants.

Assuming that there is no slip on the rod surface and that the velocity goes to zero as $r\to\infty$, the boundary conditions become

$$r=r_0 \;\Rightarrow\; v(r_0,t)=V_1\cos(\omega_1 t) \quad \text{and} \quad w(r_0,t)=V_2\cos(\omega_2 t), \quad (3.10a)$$

$$r\to\infty, \qquad v, w\to 0. \quad (3.10b)$$

Equations (3.8b) and (3.8c) can be solved together with the boundary conditions given by (3.10a) and (3.10b). Then the pressure field can be found, to within a constant, from (3.8a).

Introducing the reference velocity

$$U=\sqrt{V_1^2+V_2^2},$$

and defining the variables

$$R = \frac{r}{r_0}, \qquad \bar{t} = \frac{tU}{r_0}, \qquad \bar{v} = \frac{v}{U}, \qquad \bar{w} = \frac{w}{U},$$

we obtain the dimensionless form of equations (3.8b) and (3.8c):

$$\begin{aligned}
&\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\bar{v}}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R^2}\right) + \Gamma_1\frac{\partial}{\partial\bar{t}}\left(\frac{\partial^2\bar{v}}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R^2}\right)\\
&\quad + 2\bar{\beta}_3\left\{\frac{2}{R}\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)^3 + \left(\frac{V_2}{V_1}\right)^2\frac{2}{R}\left(\frac{\partial\bar{w}}{\partial R}\right)^2\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)\right.\\
&\quad + 3\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)^2\left(\frac{\partial^2\bar{v}}{\partial R^2} - \frac{1}{R}\frac{\partial\bar{v}}{\partial R} + \frac{\bar{v}}{R^2}\right) + 2\left(\frac{V_2}{V_1}\right)^2\left(\frac{\partial\bar{w}}{\partial R}\right)\left(\frac{\partial^2\bar{w}}{\partial R^2}\right)\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)\\
&\quad \left. + \left(\frac{V_2}{V_1}\right)^2\left(\frac{\partial^2\bar{w}}{\partial R^2}\right)\left(\frac{\partial^2\bar{v}}{\partial R^2} - \frac{1}{R}\frac{\partial\bar{v}}{\partial R} + \frac{\bar{v}}{R^2}\right)\right\} - \frac{\partial\bar{v}}{\partial\bar{t}} = 0,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\bar{w}}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{w}}{\partial R}\right) + \Gamma_1\frac{\partial}{\partial\bar{t}}\left(\frac{\partial^2\bar{w}}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{w}}{\partial R}\right) + 2\bar{\beta}_3\left\{\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)^2\left(\frac{\partial\bar{w}}{\partial R}\right)\right.\\
&\quad + \frac{1}{R}\left(\frac{V_2}{V_1}\right)\left(\frac{\partial\bar{w}}{\partial R}\right)^3 + 2\left(\frac{\partial\bar{w}}{\partial R}\right)\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)\left(\frac{\partial^2\bar{v}}{\partial R^2} - \frac{1}{R}\frac{\partial\bar{v}}{\partial R} + \frac{\bar{v}}{R^2}\right)\\
&\quad \left. + \left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)^2\left(\frac{\partial^2\bar{w}}{\partial R^2}\right) + 3\left(\frac{V_2}{V_1}\right)\left(\frac{\partial\bar{w}}{\partial R}\right)^2\left(\frac{\partial^2\bar{w}}{\partial R^2}\right)\right\} - \frac{\partial\bar{w}}{\partial\bar{t}} = 0.
\end{aligned} \tag{3.12}$$

Clearly, the dimensionless parameters governing the problem are

$$\mathrm{Re} = \frac{Ur_0}{\nu}, \qquad \Gamma_1 = \frac{\alpha_1}{\rho r_0^2}, \qquad \bar{\beta}_3 = \frac{\beta_3 U}{r_0^3\rho}, \qquad k = \frac{V_2}{V_1},$$

where Re is the Reynolds number based on the reference velocity U, Γ_1 is the absorption number and $\bar{\beta}_3$ is a parameter based on the higher viscosity β_3 which resembles the Reynolds number.

4. A PARTICULAR CASE: THE ROTATING ROD

Let us suppose for the time being that the rod simply rotates about its axis with constant angular velocity: this is a simplified version of the problem under investigation and a case that often occurs in oil drilling design.

We assume a velocity field of the kind

$$\bar{\mathbf{v}} = v(R)\mathbf{e}_\theta$$

when the boundary conditions are

$$v = 1 \quad \text{at } R = 1,$$

$$v \to 0 \quad \text{as } R \to \infty.$$

After integrating once with respect to R, the momentum equation (3.8c) in the θ direction yields

$$\frac{1}{\mathrm{Re}}F + 2\bar{\beta}_3 F^3 = \frac{C}{R^2}, \tag{4.1}$$

where

$$F = \left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)$$

and C is the undetermined integration constant.

Equivalently, the above equation can be rewritten as

$$F + \alpha F^3 = \frac{\gamma}{R^2}, \tag{4.2}$$

where

$$\alpha = 2\bar{\beta}_3 \mathrm{Re}$$

and γ is an integration constant.

Following [12], the methods to obtain the exact solution and the series solution to (4.2) will be outlined.

4.1. *Exact solution*

Equation (4.2) can be solved formally [13], the only real solution being

$$F = \sqrt[3]{\frac{\gamma}{2\alpha R^2} + \sqrt{\frac{\gamma^2}{4\alpha^2 R^4} + \frac{1}{27\alpha^3}}} + \sqrt[3]{\frac{\gamma}{2\alpha R^2} - \sqrt{\frac{\gamma^2}{4\alpha^2 R^4} + \frac{1}{27\alpha^3}}}. \tag{4.3}$$

Let us define a new variable a as

$$a = \sqrt{\frac{27}{4}\frac{\gamma}{R^2}}\sqrt{\alpha}. \tag{4.4}$$

Substituting (4.4) into (4.3), we have

$$F = \frac{1}{\sqrt{3\alpha}}\{\sqrt[3]{a + \sqrt{1 + a^2}} + \sqrt[3]{a - \sqrt{1 + a^2}}\}. \tag{4.5}$$

For small values of a, the above expression can be expanded using the binomial theorem

$$F(R, \alpha) = \frac{1}{\sqrt{3\alpha}}\left\{\frac{2a}{3} - \frac{8}{81}a^3 + \cdots\right\}. \tag{4.6a}$$

Let us assume a power series expansion for the undetermined constant γ:

$$\gamma = \gamma_0 + \alpha\gamma_1 + \alpha^2\gamma_2 + \cdots, \tag{4.6b}$$

and consider the first two terms of the Taylor expansion of F with respect to α as

$$F = F_0 + \left.\frac{\mathrm{d}F}{\mathrm{d}\alpha}\right|_{\alpha=0}\alpha + \cdots,$$

where

$$F_0 = \lim_{\alpha\to 0} F(R, \alpha).$$

Now expression (4.6a) can be rewritten as

$$F = \frac{\gamma_0}{R^2} + \left(\frac{\gamma_1}{R^2} - \frac{\gamma_0^3}{R^6}\right)\alpha - 3\left[\left(\frac{\gamma_0}{R^2}\right)^2\left(\frac{\gamma_1}{R^2} - \left(\frac{\gamma_0}{R^2}\right)^3\right) + \frac{\gamma_2}{R^2}\right]\alpha^2 + O(\alpha^3). \tag{4.7}$$

Integrating term-by-term and applying the boundary conditions

$$v(R, \alpha)|_{R=1} = 1, \qquad v(R, \alpha)|_{R\to\infty} = 0,$$

the constants γ_0, γ_1 and γ_2 are found to be

$$\gamma_0 = -2, \qquad \gamma_1 = -\tfrac{8}{3}, \qquad \gamma_2 = \tfrac{128}{15}.$$

Finally, we have

$$v(R) = \frac{1}{R} - \frac{4}{3R}\left(\frac{1}{R^4} - 1\right)\alpha + \frac{12}{R}\left(\frac{1}{5R^9} - \frac{1}{9R^5} - \frac{4}{45}\right)\alpha^2 + O(\alpha^2).$$

In order for the expansion of (4.5) to converge, the binomial theorem requires that $|a| < 1$, that is,

$$-1 < \tfrac{3}{2}\sqrt{3\alpha}\frac{\gamma}{R^2} < 1 \quad \forall R. \tag{4.8}$$

Now, after substituting into (4.8) the approximate expression of γ, that is,

$$\gamma = \gamma_0 + \alpha\gamma_1 + \alpha^2\gamma_2 = -2 - \tfrac{8}{3}\alpha + \tfrac{128}{15}\alpha^2, \tag{4.9}$$

which is obtained from its expansion in power series with respect to α, and considering that

(4.8) is always satisfied if it is true for $R = 1$, we get

$$-1 < \tfrac{3}{2}\left(\tfrac{128}{15}\alpha^2 - \tfrac{8}{3}\alpha - 2\right)\sqrt{3\alpha} < 1. \tag{4.10}$$

The numerical solution of (4.10) yields a radius of convergence of $\alpha = 0.0342$.

4.2. *Series solution*

Equation (4.2) is solved by a regular perturbation method. The function $F(R, \alpha)$ is expanded in power series about the point $\alpha = 0$ as follows:

$$F(R, \alpha) = F_0(R) + \alpha F_1(R) + \alpha^2 F_2(R) + \cdots. \tag{4.11}$$

By substitution of (4.11) and (4.6a) into equation (4.2) and after equating the coefficients of like powers of α, integrating, and applying the boundary conditions, we have

$$v(R) = \frac{1}{R} - \frac{4}{3R}\left(\frac{1}{R^4} - 1\right)\alpha + \frac{12}{R}\left(\frac{1}{5R^9} - \frac{1}{9R^5} - \frac{4}{45}\right)\alpha^2 + O(\alpha^3). \tag{4.12}$$

Note that (4.12) perfectly matches the expansion of the exact solution, as expected, since the power series expansion of a function is unique.

The general solution $v_n(R)$ can be shown to be

$$v_n(R) = -\frac{1}{2R}\sum_{p=0}^{n} \frac{a_{np}}{(1+2p)R^{4p}}, \tag{4.13}$$

where the coefficients a_{np} are given by the recurrence formula

$$a_{np} = -\sum_{m=0}^{p-1}\sum_{k=0}^{p-m-1}\sum_{i=m}^{n-p+m}\sum_{j=k}^{n-p+k+m-1} a_{im}a_{jk}a_{n-i-j-1,p-m-k-1} \tag{4.14}$$

and

$$a_{00} = -2.$$

5. PERTURBATION METHOD SOLUTION OF THE FULL PROBLEM

Let us assume a regular perturbation expansion for the velocity field around $\bar{\beta}_3 = 0$:

$$\bar{v} = \bar{v}_0 + \bar{\beta}_3\bar{v}_1 + \bar{\beta}_3^2\bar{v}_2 + \cdots, \tag{5.1}$$

$$\bar{w} = \bar{w}_0 + \bar{\beta}_3\bar{w}_1 + \bar{\beta}_3^2\bar{w}_2 + \cdots. \tag{5.2}$$

After substitution in the governing equations, by equating the coefficients of like powers of $\bar{\beta}_3$ we have, for the terms of $O(1)$,

$$\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\bar{v}_0}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{v}_0}{\partial R} - \frac{\bar{v}_0}{R^2}\right) + \Gamma_1\frac{\partial}{\partial\bar{t}}\left(\frac{\partial^2\bar{v}_0}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{v}_0}{\partial R} - \frac{\bar{v}_0}{R^2}\right) - \frac{\partial\bar{v}_0}{\partial\bar{t}} = 0, \tag{5.3}$$

$$\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\bar{w}_0}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{w}_0}{\partial R}\right) + \Gamma_1\frac{\partial}{\partial\bar{t}}\left(\frac{\partial^2\bar{w}_0}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{w}_0}{\partial R}\right) - \frac{\partial\bar{w}_0}{\partial\bar{t}} = 0, \tag{5.4}$$

with boundary conditions

$$\bar{v} = \frac{V_1}{\sqrt{V_1^2 + V_2^2}}\cos\Omega_1\bar{t} \quad \text{and} \quad \bar{w} = \frac{V_2}{\sqrt{V_1^2 + V_2^2}}\cos\Omega_2\bar{t} \text{ at } R = 1.$$

Also,

$$\bar{v}_0, \bar{w}_0 \to 0 \quad \text{as } R \to \infty.$$

For the terms of $O(\bar{\beta}_3)$,

$$\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\bar{v}_1}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{v}_1}{\partial R} - \frac{\bar{v}_1}{R^2}\right) + \Gamma_1\frac{\partial}{\partial\bar{t}}\left(\frac{\partial^2\bar{v}_1}{\partial R^2} + \frac{1}{R}\frac{\partial\bar{v}_1}{\partial R} - \frac{\bar{v}_1}{R^2}\right)$$

$$+ 2\left\{\frac{2}{R}\left(\frac{\partial\bar{v}_0}{\partial R} - \frac{\bar{v}_0}{R}\right)^3 + \left(\frac{V_2}{V_1}\right)^2\frac{2}{R}\left(\frac{\partial\bar{w}_0}{\partial R}\right)^2\left(\frac{\partial\bar{v}_0}{\partial R} - \frac{\bar{v}_0}{R}\right)\right.$$

$$+3\left(\frac{\partial \bar{v}_0}{\partial R}-\frac{\bar{v}_0}{R}\right)^2\left(\frac{\partial^2 \bar{v}_0}{\partial R^2}-\frac{1}{R}\frac{\partial \bar{v}_0}{\partial R}+\frac{\bar{v}_0}{R^2}\right)+2\left(\frac{V_2}{V_1}\right)^2\left(\frac{\partial \bar{w}_0}{\partial R}\right)\left(\frac{\partial^2 \bar{w}_0}{\partial R^2}\right)\left(\frac{\partial \bar{v}_0}{\partial R}-\frac{\bar{v}_0}{R}\right)$$

$$+\left(\frac{V_2}{V_1}\right)^2\left(\frac{\partial^2 \bar{w}_0}{\partial R^2}\right)\left(\frac{\partial^2 \bar{v}_0}{\partial R^2}-\frac{1}{R}\frac{\partial \bar{v}_0}{\partial R}+\frac{\bar{v}_0}{R^2}\right)\Bigg\}-\frac{\partial \bar{v}_1}{\partial \bar{t}}=0, \tag{5.5}$$

$$\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \bar{w}_1}{\partial R^2}+\frac{1}{R}\frac{\partial \bar{w}_1}{\partial R}\right)+\Gamma_1\frac{\partial}{\partial \bar{t}}\left(\frac{\partial^2 \bar{w}_1}{\partial R^2}+\frac{1}{R}\frac{\partial \bar{w}_1}{\partial R}\right)+2\Bigg\{\left(\frac{\partial \bar{v}_0}{\partial R}-\frac{\bar{v}_0}{R}\right)^2\left(\frac{\partial \bar{w}_0}{\partial R}\right)$$

$$+\frac{1}{R}\left(\frac{V_2}{V_1}\right)\left(\frac{\partial \bar{w}_0}{\partial R}\right)^3+2\left(\frac{\partial \bar{w}_0}{\partial R}\right)\left(\frac{\partial \bar{v}_0}{\partial R}-\frac{\bar{v}_0}{R}\right)\left(\frac{\partial^2 \bar{v}_0}{\partial R^2}-\frac{1}{R}\frac{\partial \bar{v}_0}{\partial R}+\frac{\bar{v}_0}{R^2}\right)$$

$$+\left(\frac{\partial \bar{v}_0}{\partial R}-\frac{\bar{v}_0}{R}\right)^2\left(\frac{\partial^2 \bar{w}_0}{\partial R^2}\right)+3\left(\frac{V_2}{V_1}\right)\left(\frac{\partial \bar{w}_0}{\partial R}\right)^2\left(\frac{\partial^2 \bar{w}_0}{\partial R^2}\right)\Bigg\}-\frac{\partial \bar{w}_1}{\partial \bar{t}}=0. \tag{5.6}$$

The exact solution to (5.1) and (5.2) has been found by Rajagopal [7] and is given here in dimensionless form:

$$\bar{v}(R,\bar{t})=\mathrm{Re}\left\{\frac{K_1\left[\left(\dfrac{1}{(1/\mathrm{i}\,\mathrm{Re}\,\Omega_1)+\Gamma_1}\right)^{1/2}R\right]}{K_1\left[\left(\dfrac{1}{(1/\mathrm{i}\,\mathrm{Re}\,\Omega_1)+\Gamma_1}\right)^{1/2}\right]}\frac{V_1}{\sqrt{V_1^2+V_2^2}}\mathrm{e}^{\mathrm{i}\Omega_1\bar{t}}\right\}, \tag{5.7}$$

$$\bar{w}(R,\bar{t})=\mathrm{Re}\left\{\frac{K_0\left[\left(\dfrac{1}{(1/\mathrm{i}\,\mathrm{Re}\,\Omega_2)+\Gamma_1}\right)^{1/2}R\right]}{K_0\left[\left(\dfrac{1}{(1/\mathrm{i}\,\mathrm{Re}\,\Omega_2)+\Gamma_1}\right)^{1/2}\right]}\frac{V_2}{\sqrt{V_1^2+V_2^2}}\mathrm{e}^{\mathrm{i}\Omega_2\bar{t}}\right\}, \tag{5.8}$$

where Re denotes the real part of the complex number which follows and

$$\Omega_1=\frac{\omega_1 r_0}{U},\qquad \Omega_2=\frac{\omega_2 r_0}{U}.$$

These expressions are to be used in (5.5) and (5.6), which can be solved numerically.

6. DRAG, AXIAL SHEAR FORCE AND TORQUE

Recall the definition of Stokes drag in a fluid [14]:

$$D=\int_{\partial\wp}\mathbf{t}\cdot\mathbf{n}\,\mathrm{d}a, \tag{6.1}$$

where $\mathbf{t}$ is the traction on the boundary of the body $\partial\wp$ and $\mathbf{n}$ is the unit vector in the direction of the uniform velocity at infinity. Rather than in the Stokes drag, we are interested in computing the axial shear force and the torque per unit length that are required to produce the prescribed oscillations of the rod. This information is of great help in the design of any oil drilling machinery or equipment.

The torque and the axial force per unit length are given, respectively, by

$$\mathbf{M}=2\pi r_0^2\tau_{r\theta}, \tag{6.2}$$

$$\mathbf{F}=2\pi r_0\tau_{rz}, \tag{6.3}$$

where

$$\tau_{r\theta}=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)+2\beta_3\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\left[\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2+\left(\frac{\partial w}{\partial r}\right)^2\right], \tag{6.4}$$

$$\tau_{rz}=\mu\frac{\partial w}{\partial r}+\alpha_1\frac{\partial}{\partial t}\frac{\partial w}{\partial r}+2\beta_3\frac{\partial w}{\partial r}\left[\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)^2+\left(\frac{\partial w}{\partial r}\right)^2\right]. \tag{6.5}$$

Let us non-dimensionalize (6.3) and (6.4):

$$\bar{\mathbf{M}} = \frac{\mathbf{M}}{2\pi\rho U^2 r_0^2} = \frac{1}{\mathrm{Re}}\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right) + \Gamma_1\frac{\partial}{\partial\bar{t}}\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right) + 2\bar{\beta}_3\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)\left[\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)^2 + \left(\frac{\partial\bar{w}}{\partial R}\right)^2\right], \tag{6.6}$$

$$\bar{\mathbf{F}} = \frac{\mathbf{F}}{2\pi\rho U^2 r_0} = \frac{1}{\mathrm{Re}}\frac{\partial\bar{w}}{\partial R} + \Gamma_1\frac{\partial}{\partial\bar{t}}\frac{\partial\bar{w}}{\partial R} + 2\bar{\beta}_3\frac{\partial\bar{w}}{\partial R}\left[\left(\frac{\partial\bar{v}}{\partial R} - \frac{\bar{v}}{R}\right)^2 + \left(\frac{\partial\bar{w}}{\partial R}\right)^2\right]. \tag{6.7}$$

The overbar denotes dimensionless quantities.

7. STRESS POWER

The stress power in a third-grade fluid is given by [15]:

$$\mathbf{T}\cdot\mathbf{L} = \frac{\mu}{2}|\mathbf{A}_1|^2 + \frac{1}{4}\alpha_1\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{A}_1|^2 + \frac{\alpha_1+\alpha_2}{2}\operatorname{tr}\mathbf{A}_1^3 + \frac{1}{2}\beta_3|\mathbf{A}_1|^4, \tag{7.1}$$

where $|\mathbf{A}_1|$ is defined as the trace norm of the tensor $\mathbf{A}_1$.

In our case, after non-dimensionalization, (7.1) reduces to

$$\overline{\mathbf{T}\cdot\mathbf{L}} = \frac{\mathbf{T}\cdot\mathbf{L}}{\rho U^3/r_0} = \frac{1}{2\mathrm{Re}}|\bar{\mathbf{A}}_1|^2 + \frac{1}{4}\Gamma_1\frac{\partial}{\partial\bar{t}}|\bar{\mathbf{A}}_1|^2 + \frac{1}{2}\bar{\beta}_3|\bar{\mathbf{A}}_1|^4. \tag{7.2}$$

Integration of (7.2) through the whole domain yields

$$\int_V \frac{\mathbf{T}\cdot\mathbf{L}}{2\pi\rho U^3 r_0^2}\,\mathrm{d}v = \int_1^\infty\left(\frac{1}{2\mathrm{Re}}|\bar{\mathbf{A}}_1|^2 + \frac{1}{4}\Gamma_1\frac{\partial}{\partial\bar{t}}|\bar{\mathbf{A}}_1|^2 + \frac{1}{2}\bar{\beta}_3|\bar{\mathbf{A}}_1|^4\right)R\,\mathrm{d}R. \tag{7.3}$$

8. NUMERICAL ALGORITHM OF THE FULL PROBLEM

The finite difference method and a regular mesh are used for the numerical solution. The transformation $S = 1/R$ is applied for transforming the infinite physical domain to a finite calculation domain, that is:

$$R \subseteq [R_0, \infty], \qquad S \subseteq [S_0, 0],$$

$$R = \frac{1}{S}, \quad \frac{\partial}{\partial R} = -S^2\frac{\partial}{\partial S}, \quad \frac{\partial^2}{\partial R^2} = S^4\frac{\partial^2}{\partial S^2} + 2S^3\frac{\partial}{\partial S}.$$

The Crank–Nicholson method is employed in this time-dependent problem. Due to the highly non-linear terms in the governing equations, the convergence of the numerical solution depends on the choice of the initial condition, that is, on the velocity distribution at time $t = 0$. The exact solution to the problem under investigation for second-order fluid is available [7] and is used as an initial condition.

9. RESULTS AND DISCUSSION

Figures 1 and 2 show how the velocity profile varies with β_3: the velocity is plotted at equal intervals of time in a half cycle. It is clear that, as the higher-order viscosity increases, the velocity gradient becomes steeper and loses monotonicity. The dynamical boundary layer does not seem to be significantly affected by β_3 because the elastic forces still dominate the viscous ones.

Torque and force vs time are reported for various values of β_3 (Figs 3 and 4); as expected their maximum increases with β_3. The curves representing torque and force are similar because the periods of the longitudinal and torsional oscillations are taken as equal in the actual computations.

Figure 5 confirms, in the case under investigation, what Rajagopal [2] anticipated for a different problem: the stress power is locally negative at some instants of time as well as in the whole domain (Fig. 6). The smaller the β_3, the longer the stress power remains negative.

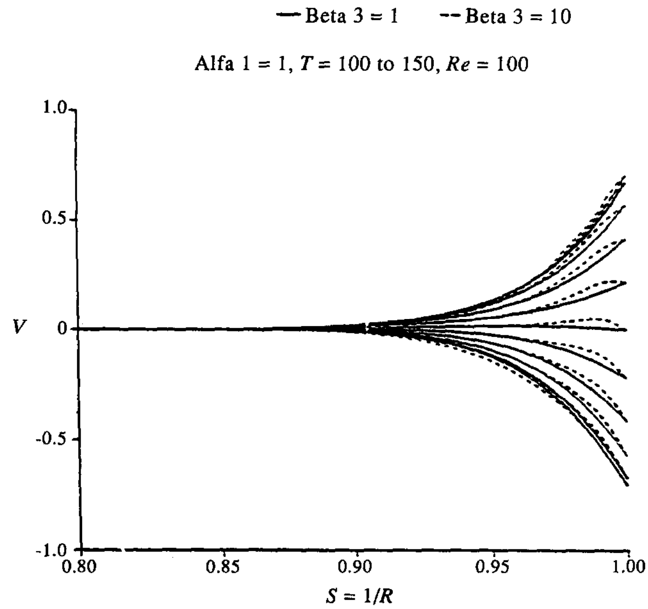

Fig. 1.

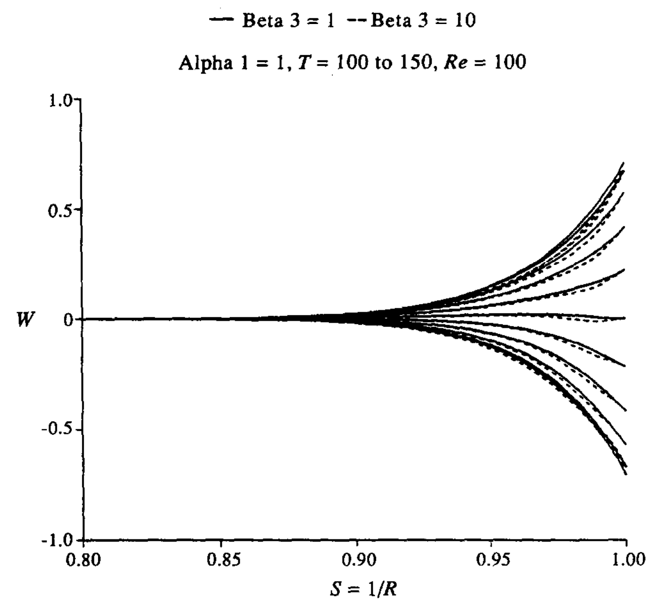

Fig. 2.

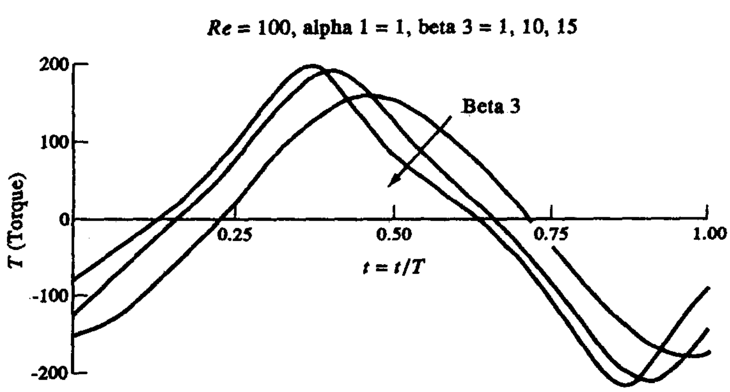

Fig. 3.

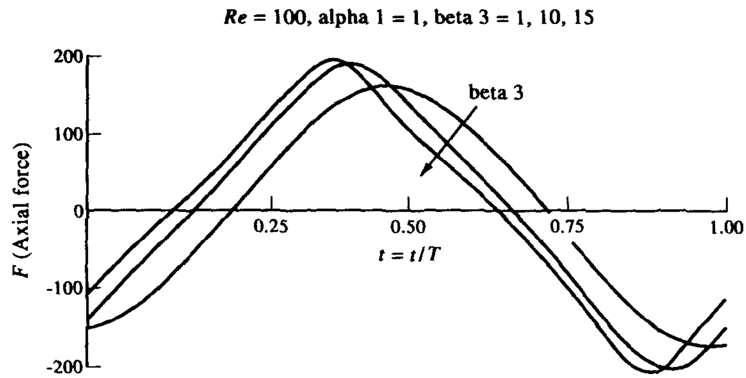

Fig. 4.

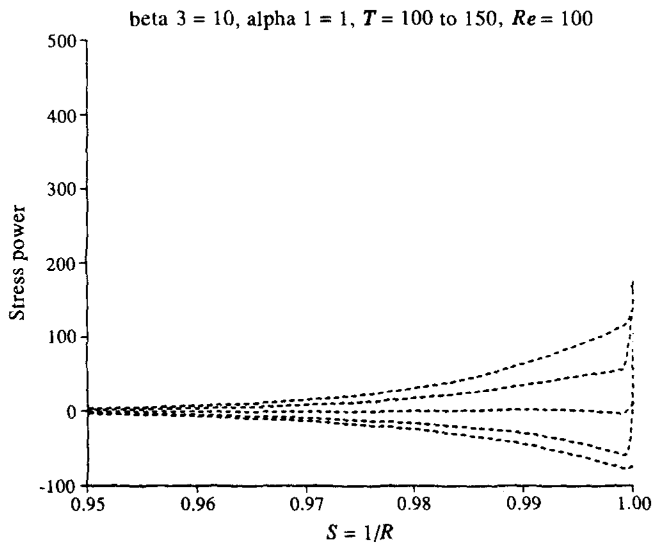

Fig. 5.

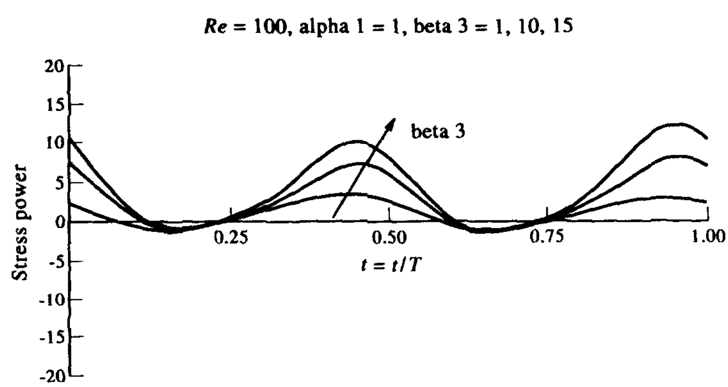

Fig. 6.

A comparison between Figs 5 and 7, which represent the stress power in a power-law fluid, helps to understand that the elastic effect is responsible for this phenomenon.

The influence of the elastic forces on the velocity can be fully appreciated by examining the velocity profile for a purely viscous fluid (Figs 8 and 9) and a viscoelastic fluid (Figs 1 and 2). The wavy profile typical of purely viscous fluids is substituted by a fan profile in a material in which elastic effects are present.

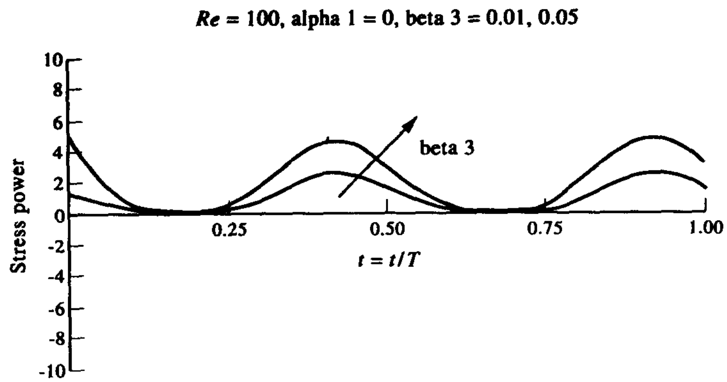

Fig. 7.

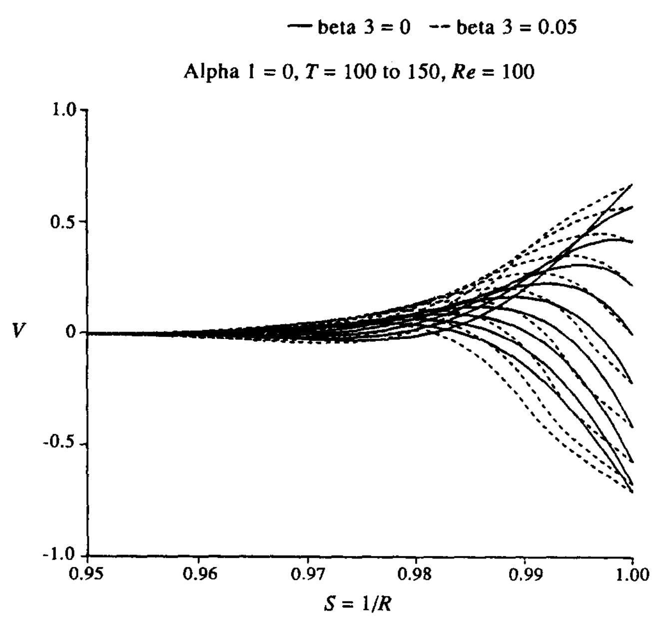

Fig. 8.

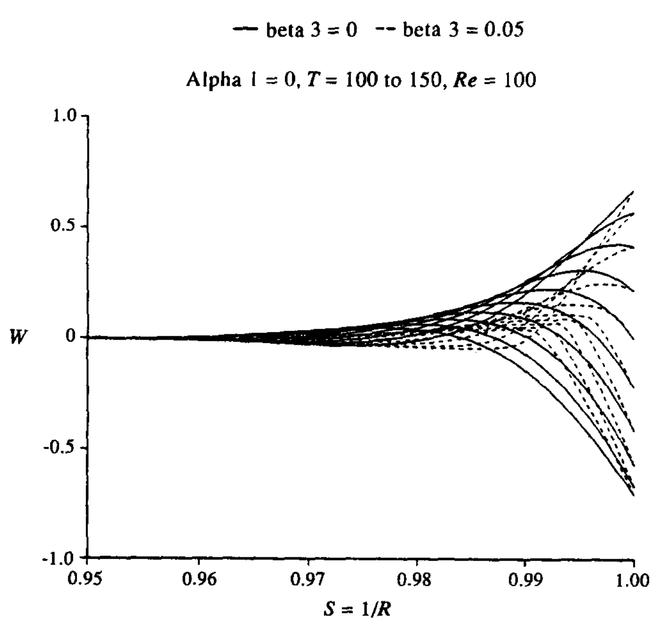

Fig. 9.

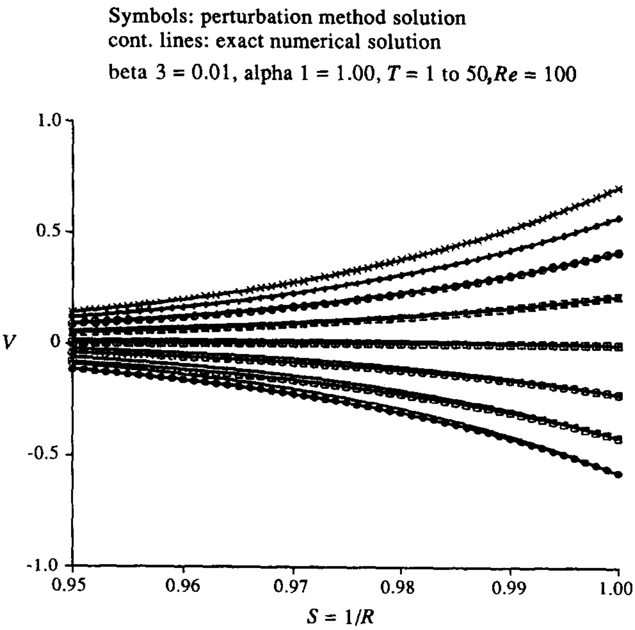

Fig. 10.

Figure 10 reports the perturbation method results, which are compared with the exact solution for a particular case.

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