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Research Article

On spectra of *n*-hyponormal operators in Hilbert space

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Abstract

Many practical applications of Mathematics rely on results in operator theory. In this paper we focus on the characterization of the spectrum of a hyponormal operator and the spectrum of its adjoint. Considering an atomic quantum mechanical system, if *A* is an operator of an atom, then the differences of the various eigenvalues of *A* are the amounts of energy emitted by the atom as it undergoes transitions. These amounts are seen in the form of electromagnetic waves, which constitute the optical spectrum of the report. The main objective will be finding a formal evaluation of the spectra of hyponormals and the spectrum of its adjoint. Emphasis will also be on the resultant spectra of similar operators to find any relationships.

Keywords: Operator; *n*-Power-hyponormal; Eigenvalues; Eigenvectors; Conjugates; Transpose; Adjoint; Spectrum.

Introduction

The term spectrum arises from the following physical considerations. If a physical quantity (like position, momentum or energy) is represented by an operator *A*, and is measured in an experiment, then the result of the measurement is one of the eigenvalues of *A* [1]. The amounts are seen in the form of electromagnetic waves, which constitute the optical spectrum. This accounts for such series observed in atomic analysis such as the Balmer series and Lyman series [2].

Throughout this paper, *B(H)* denotes the algebra of all bounded linear operators acting on a complex Hilbert space, H. An operator is said to be an n-normal operator if $T^n T^* = T^*T^n$; normal if $TT^* = T^*T$ (it is clear that a bounded normal operator is an n-normal operator for any n); self adjoint if $T^* = T$; positive if $T^* = T$ and $\langle Tx, x\rangle \geq 0$ for all x, and semi-normal if $T^2 = T^{*2}$; projection if $T^2 = T = T^*$ [6]. For an operator T ∈ H, if $\|Tx\| = \|x\|$ for all x ∈ H (or equivalently $T^*T = I$), then T is called an isometry. T is called unitary if $TT^* = T^*T = I$. An operator T on H is called hyponormal if $TT^* \leq T^*T$ [4, 5, 6]. We present a general case for bounded self-adjoint operators [7]. This generalization is not merely a heuristic desire: infinite dimensions are inescapable. Indeed, mathematical physics is necessarily done in an infinite dimensional setting. Moreover, quantum theory requires the careful study of functions of operators on these spaces [8]. Though it may seem abstract at first, an example of a function of operators is encountered with systems of linear Ordinary Differential Equations (ODEs). Given a system of ordinary linear differential equations of the form x(t) = Ax(t) where A is a constant matrix, the solution is given by $x(t) = e^{tA} \times 0$. This is an instance of the matrix exponential, an operation that is well defined for finite dimensions. Yet, quantum mechanics demands that we are able to define objects like this for any operator. In particular, the time evolution of a quantum mechanical state, r is expressed by conjugating the state by $\exp(itH)$ where H is the Hamiltonian of the system [9].

We have limited our study to the hyponormal operators. A function f is defined to

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be a relation, such that if $(x, y) \in f$, *and* $(x, z) \in f$, then $y = z$. Four other terms that may be used for a function are: map, mapping, operator or transformation [10]. A function is a certain set of ordered pairs [11], and as such it can actually be represented graphically. If y is a function, and $(x, y) \in f$, then we write $y = f(x)$. We say y is the value of *f* at x, or that y is the image of x under *f*. $f : X \rightarrow Y$ implies *f* takes elements from space X into space Y. Now X is the domain, Y is the range. X and Y can be topological spaces. The operator is thus a mapping from one vector space to another or from one module to another. An example of a function of operators is encountered with systems of linear ordinary differential equations of the form $Ax(t) = x(t)$, where A is a constant matrix; and the solution is given by $x(t) = e^{tA} \times 0$.

Operators can be represented by matrices. Let H be Hilbert space, and $A \in B(H)$, the set of bounded linear operators on H. We focus on the self adjoint operators. An operator A is self adjoint if, as a matrix, $A = A^*$, where A^* denotes the conjugate transpose of A. In infinite dimensional space, this definition does not apply directly, but relies on the notion of an adjoint operator in a Hilbert space [12].

Diagonalization is one of the most important topics in linear algebra [13]. Unfortunately, it only works on finite dimensional vector spaces, where linear operators can be represented by finite matrices. Eventually, one encounters infinite dimensional vector spaces (spaces of sequences, for instance), where linear operators can be thought of as infinite matrices. Extending the idea of diagonalization to these operators requires some new machinery [14].

Let H be a Hilbert space and $A \in B(H)$, the set of bounded linear operators on H . In particular, in this exposition, we focused on self-adjoint operators. In finite dimensions, an operator A is called self-adjoint if, as a matrix, $A = A^*$, where A^* denotes the conjugate transpose of A. In infinite dimensional space, this definition does not apply directly. We first need the notion of an adjoint operator in a Hilbert space. We begin by stating a result that will be used several times in this exposition [15].

Now, let $\lambda \in C$ be such that $|\lambda| > \|T\|$. Then, $\exists\ \delta \in R$, such that $\|\lambda\| > \delta > \| T \|$. This means that $\forall\, x \in H$, $\| Tx \| \leq \| T \| < \| \delta x \| < \| \lambda x \|$; And thus, $\forall\, x$, $0 < \| (\lambda I - T) - 1x \| < \infty$, so that $\lambda \in \rho(T)$. As mentioned, we can represent differential operators by finite dimensional matrices to solve ODEs. Now we will consider using the same representations to determine the spectrum (eigenvalues) of the operators. In short, we approximate the spectrum of the infinite-dimensional operator by computing the eigenvalues of its matrix approximation. Let $A \in C^{n\times n}$. We denote its spectrum (eigenvalues) by $\sigma(A)$, i.e. $\lambda \in \sigma(A)$, if there exists v such that $Av = \lambda v$. Or the determinant vanishes; $det\,(A - \lambda I) = 0$. Or the operator is not invertible. (This translates to operators in Banach Spaces) [16].

Research methodology

Definition 1.1

T ∈ B(H) is called an n-power-hyponormal operators if $T^nT^* \leq T^*T^n$. We observe that, this new class includes all normal, all n-normal and all hyponormal operators. This makes the hyponormals a fine class to use in representing other classes of operators.

Definition 1.2

Let T ∈ B(H). *For* $y \in H$, the map $x \rightarrow \langle y|Tx\rangle$ defines a bounded linear operator. Riesz's representation theorem for Hilbert spaces then tells us that $\exists\ z \in H$, such that $\phi(x) = \langle y|Tx\rangle = \langle z|x\rangle$. We now write $T^*(y) = z$ and define the adjoint T^* this way [13]. An operator $A \in B(H)$ is said to be self-adjoint if $\langle Ax|y\rangle = \langle x|Ay\rangle$ for all $x, y \in H$, that is if $A = A^*$ with respect to our definition of the adjoint above [9].

Definition 1.3:

λ is an eigenvalue of A if there exists $v \neq 0$, $v \in H$ such that $Av = \lambda v$. Equivalently, λ is an eigenvalue if and only if $(A - \lambda I)$ is not injective.

Remark 1.4

Several important properties of self-adjoint operators follow directly from our definition. First, the eigenvalues of a self-adjoint operator, A, are real. Indeed, let $Av = \lambda v$. Then $\lambda\langle v|v\rangle = \langle Av|v\rangle = \langle v|Av\rangle = \lambda\langle v|v\rangle$. So $\lambda = \bar{l}\lambda$. Moreover, if $Av = \lambda v$, $Au = \mu u$ then $\lambda\langle v|u\rangle = \langle Av|u\rangle = \langle v|Au\rangle = \mu\langle v|u\rangle$. Since $\lambda \neq \mu = \mu$ we conclude that $\langle v|u\rangle = 0$, which tells us that the eigen spaces of A corresponding to different eigenvalues are orthogonal. These two simple facts are not only reassuring, but crucial for the study of quantum mechanical systems. In fact, for a quantum system, the Hamiltonian is a self-adjoint operator whose eigenvalues correspond to the energy levels of the bound states of the system [2].

Definition 1.5:

The resolvent set of T, $\rho(T)$ is the set of all complex numbers λ such that $R\lambda(T) := (\lambda I - T)^{-1}$ is a bijection with a bounded inverse. The spectrum of T, $\sigma(T)$ is then given by $C\backslash\rho(T)$.

Remark 1.6

In general, the spectrum of a linear operator T is comprised of two disjoint components:
1. The set of eigenvalues, (now called the point spectrum).
2. The remaining part which is called the continuous spectrum.

We also note that: a. the eigenvalues of a real matrix need not be real numbers. For example; the characteristic polynomial of the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is x^2+1 so the eigenvalues of A are the non-real complex roots $\lambda = i$ *and* $\lambda = -i$ Furthermore, the spectral radius of T is defined by $\sigma(T^*) = sup|\lambda| : \lambda \in \sigma(T)$. The point spectrum and the approximate point spectrum of an operator T are parts of the spectrum. They are denoted by $\sigma p(T)$ and $\sigma ap(T)$ respectively. The point spectrum of T is, by definition, the set of all scalars λ such that $(T - \lambda) \neq (0)$. Furthermore, $\sigma ap(T)$ consists of all $\lambda \in C$ for which there is a sequence h_n, $n \in H$ such that $\|h_n\| = 1\ \forall n$ and $\|(T - \lambda)hn\| \to 0$ as $n \to \infty$. Mendelson in [20] proves an important result about the spectrum of T, namely that: The spectrum of a bounded linear operator is a closed and bounded subset of C. In fact, $\sigma(T) \subseteq z \in C : |z| \leq \| T \|$.

Results and discussion

There are many conditions for a linear bounded operator on a Hilbert space to be normal [11]. Here we characterize hyponormal operators with closed ranges. The following lemmas are central.

Lemma 2.1

Let M ∈ B(H) be self adjoint and let M have the decomposition (with respect to the orthogonal sum of subspaces of H):
$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$, such that A has a closed range. Then $M \geq 0$ if and only if the following hold:

I. $A \geq 0$;
II. $AA^{-1}B = B$
III. $C - B^*A^{-1}B \geq 0$.

We also need the following elementary result:

Lemma 2.2

Let A ∈ B (H). If A and AA* + A*A have closed ranges, then:
$(AA^* + A^*A)(AA^* + A^*A)^{-1}AA^* = AA^*$, i.e. $\mathcal{R}(AA^*) \subset \mathcal{R}(AA^* + A^*A)$.

Proof:

Let $M = \begin{bmatrix} AA^* + A^*A & AA^* \\ AA^* & AA^* \end{bmatrix}$, as A is a closed range operator, so is AA^*. Since $AA^* \geq 0$, $AA^*(AA^*)^{-1}AA^* = AA^*$ and $AA^* + A^*A - AA^*(AA^*)^{-1} = A^*A \geq 0$, by lemma above. It follows that $M \geq 0$. Further application of lemma 3.1 obtains $(AA^* + A^*A)(AA^* + A^*A)^{-1}AA^* = AA^*$ is satisfied [11].

Theorem 2.3

Let A and AA* + A*A have closed ranges. Then the following statements are equivalent:

1. A is hyponormal
2. $2AA^*AA^* + A^*A)^{-1}AA^* \leq AA^*$.

Proof:

(1) ⇒ (2): Let A be hyponormal, i.e. $A^*A \geq AA^*$. Consider the matrix $M = \begin{bmatrix} AA^* + A^*A & AA^* \\ AA^* & \frac{1}{2}AA^* \end{bmatrix}$
Since $\frac{1}{2}AA^* \geq 0$,
$\frac{1}{2}AA^*(½AA^*)^{-1}AA^* = AA^*$, and $AA^* + A^*A - AA^*(\frac{1}{2}AA^*)^{-1}AA^* = A^*A - AA^* \geq 0$.
By lemma 2.1, we get that $M \geq 0$. Hence we get that $½AA^* - AA^*(AA^* + A^*A)^{-1}AA^* \geq 0$, which satisfies (2).
(2) ⇒ (1): Suppose that (2) holds. By lemma 2.2 we have that:
$(AA^* + A^*A) \geq 0$, $(AA^* + A^*A)(AA^* + A^*A)^{-1}AA^* = AA^*$. $½AA^* - AA^*(AA^* + A^*A)^{-1}AA^* \geq 0$.

And by lemma 2.1 , we conclude that the operator M = $\begin{bmatrix} AA^* + A^*A & AA^* \\ AA^* & \frac{1}{2}AA^* \end{bmatrix}$ is non-negative. Applying lemma 2.1 to M, using opposite blocks, we conclude that $A^*A \geq AA^*$, i.e A is hyponormal.

Theorem 2.4

An operator S is said to be similar to an operator T in case there exists an invertible operator A such that S = $A^{-1}TA$. Here, all the operators will relate to a Hilbert space H. Sheth I. asserts by lemma that if an operator A is similar to an operator B, the S is bounded below if and only if B is bounded below. In other words, if A and B are similar, then $\sigma_{ap}(A) = \sigma_{ap}(B)$

Proof:

Let A= $T^{-1}BT$ for an invertible operator T. Now if B is bounded below, then $B^*B = \lambda I$ for some constant $\lambda > 0$. Since T is invertible, there exist constants $\beta > 0$ and $\gamma > 0$ such that $T^*T \geq \beta I$ and $(TT^*)^{-1} = T^{*-1}T^{-1} \geq \gamma I$. Now A^*A =

$T^*B^*T^{*-1}T^{-1}BT = (BT)^*T^{*-1}T^{-1}BT \geq (BT)^*\gamma IBT = \gamma T^* B^*BT \geq \gamma T^* \lambda IT = \lambda\gamma T^*T > \lambda\beta\gamma I$

i.e A is bounded below.

Since the process above is reversible, the stated result follows. The relation $\sigma_{ap}(A) = \sigma_{ap}(B)$ follows from the following two observations:

i. If A is similar to B then $A - zI$ is similar to $B - zI$ for all complex numbers z.
ii. $Z \in \sigma_{ap}(A)$ *iff* $(A - zi)$ is bounded below [1].

Remark 2.5

Consider the system of first order, linear ODEs. These can be written using matrices as $y' = Ay$.

We have that solutions to the linear ODEs have the form e^{rt}, hence $y_1 = e^{\lambda t}a$ and $y_2 = e^{\lambda t}b$. Writing in vector notation, $y = e^{\lambda t}\begin{pmatrix} a \\ b \end{pmatrix} = e^{\lambda t}x$. Here λ is the eigenvalue and x is the eigenvector. To find a solution of this form, we evaluate $\frac{d}{dt}e^{\lambda t}x = \lambda e^{\lambda t}x$. $Ae^{\lambda t}x = e^{\lambda t}Ax$. Here A is the differential $\frac{d}{dt}$ If there is a solution of this form, it satisfies the equation $\lambda e^{\lambda t}x = e^{\lambda t}Ax$, and because $e^{\lambda t}$ is never zero, we can cancel it from both sides, and end up with the central equation for eigenvalues and eigenvectors: $\lambda x = Ax$. A non-zero vector x, is an eigenvector if there is a number λ such that $Ax = \lambda x$. The scalar λ is the eigenvalue. Since it is true that $A.0 = \lambda.0$ for any λ, we require that an eigenvector must be a non-zero vector, and an eigenvalue must correspond to a non-zero vector. The scalar value λ can however be any real or complex number, including zero. The equation $Ax = \lambda x$ implies that we are looking for a vector x such that x and Ax point in the same direction. But the length can change, and is scaled by λ. The set of all such λ forms the spectrum of A. We seek to find λ in the equation $Ax = \lambda x \Rightarrow Ax - \lambda x = 0$. Thus $(A - \lambda I)x = 0$.

To satisfy our conditions for A, being hyponormal, then $(A - \lambda I)$ must be a singular matrix, i.e. have a determinant = 0. So to find λ, we solve $\det(A - \lambda I) = 0$ $A - \lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$. The determinant is a polynomial in λ. $\det(A - \lambda I) = \lambda^2 - \frac{(a+d)\lambda}{tr(A)} + \frac{(ad-bc)}{\det A} = 0$. This polynomial is the characteristic polynomial. In general, an $n \times n$ matrix would have a corresponding n^{th} degree polynomial. This polynomial encodes a lot of information: The polynomial always has n roots. These roots can be real or complex numbers. There are several observations that can be made about eigenvalues:

i. The sum of the eigenvalues is equal to the sum of diagonal entries; of the operator matrix. This is called the trace, denoted tr(A). For an $n\times n$ matrix, with $\lambda_1, \lambda_2 \ldots \lambda n$ as eigenvalues, then $\lambda_1+\lambda_2+\ldots+\lambda_n = tr(A)$.

ii. The constant term (the coefficient of λ^0) is the determinant of A.

iii. The coefficient of λ^{n-1} is the trace of A.

iv. The product of the eigenvalues is equal to the det (A); $\lambda_1.\lambda_2\ldots\lambda_n = det(A)$.

v. The roots of this polynomial are the eigenvalues of A.

vi. The other coefficients of this polynomial are more complicated invariants of the matrix A.

Remark 2.6

For an $n \times n$ matrix, we usually obtain n solutions to the homogeneous system of equations. We obtain the general solution by taking linear combinations of these n solutions. The complete solution for any system of two first order ODEs has the form $y = c_1e^{\lambda_1 t}x_1 + c_2e^{\lambda_2 t}x_2$ where c_1 and c_2 are constant parameters that can be determined from initial conditions, y_1 (0) and $y_2(0)$. It makes sense to multiply by this parameter because when we have an eigenvector, we actually have an entire line of eigenvectors, and this line of eigenvectors gives us a line of solutions; the spectrum.

Remark 2.7

This can be done by looking at the matrix and its properties, particularly the diagonal, columns, null space, relations among the columns, trace ... So we can construct a matrix with prescribed eigenvalues. Here we rely on the eigenvalue algorithm; say D = a diagonal matrix, XDX^{-1}, $v = Xu$.

Now we can look for the matrix that has these eigenvalues and eigenvectors as its spectrum. For a matrix obtained by similarity relation, XAX^{-1}. In general we place the eigenvalues along the diagonal and put this diagonal matrix into a similarity relation, and this will guarantee that we have or will get the eigenvalues that we want. So we have to choose X properly. $v = Xu$. So, whatever X we choose, the corresponding v will be the columns of the matrix X. We need to establish if this matrix is unique; and this will rely on the eigenvalue decomposition of the matrix.

Theorem 2.8

If A is an $n \times n$ matrix, then the following are equivalent:

a. A is diagonalizable.

b. A has n linearly independent eigenvectors.

Proof:

$a \Rightarrow b$ Since A is assumed diagonalizable, there is an invertible matrix P such that:

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \cdots & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & - & \lambda_n p_{1n} \\ - & - & - & - \\ - & - & - & - \\ \lambda_1 p_{n1} & - & - & \lambda_n p_{nn} \end{pmatrix}$$

If we now let p_1, p_2 ...denote the column vectors of P, the AP have $\lambda_1 p_1$, $\lambda_2 p_2$, ..., $\lambda_n p_n$ as their successive columns. Therefore $Ap_1 = \lambda_1 p_1$, $Ap_2 = \lambda_2 p_2$, ...$Ap_n = \lambda_n p_n$. Since P is invertible, $p_1 ... p_n$ are linearly independent. Thus A has n linearly independent vectors. $b \Rightarrow a$ Assume A has n linearly independent eigenvectors $p_1 ... p_n$, with corresponding eigenvalues $\lambda_1 ... \lambda_n$, and let $P = \begin{pmatrix} p_{11} & p_{12} & - & p_{1n} \\ - & - & - & - \\ - & - & - & - \\ p_{n1} & - & - & p_{nn} \end{pmatrix}$ be the matrix whose column vectors are $p_1 ... p_n$. Then the product $AP = AP_1, AP_2, ...AP_n$. But $AP_1 = \lambda P_1$ etc.

So that

$$\begin{pmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & - & \lambda_n p_{1n} \\ - & - & - & - \\ - & - & - & - \\ \lambda_1 p_{n1} & - & - & \lambda_n p_{nn} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & - & p_{1n} \\ - & - & - & - \\ - & - & - & - \\ p_{n1} & - & - & p_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & \lambda_n \end{pmatrix}$$

$= PD$,

where D is the diagonal matrix having the eigenvalues $\lambda_1 ... \lambda_n$ on the main diagonal. Thus $P^{-1}AP = D$; A is diagonalizable. The general idea is to first diagonalize the matrix A, that is find an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix, and $A = PDP^{-1}$. Squaring A would then yield $A^2 = (PDP^{-1})(PDP^1) = PD^2P^1$ etc.

If A is an $n \times n$ matrix, a number λ is called an eigenvalue of A if $Ax = \lambda x$ for some $x \neq 0$. x is called an eigenvector corresponding to the eigenvalue λ. If $P^{-1}AP = D$, *then* $AP = PD$. If $D = diag[\lambda_1 ... \lambda_n]$ then AP = PD becomes $A[x_1, x_2 ... x_n] = [x_1, x_2 ... x_n] \begin{pmatrix} \lambda_1 & 0 & - & 0 \\ - & \lambda_2 & - & - \\ - & - & - & - \\ 0 & 0 & - & \lambda_n \end{pmatrix}$

$(Ax_1, Ax_2, ... Ax_n) = (\lambda_1 x_1, \lambda_2 x_2, ... \lambda_n x_n)$

$\therefore Ax_1 = \lambda_1 x_1; Ax_2 = \lambda_2 x_2; ...; Ax_n = \lambda_n x_n$

$\Rightarrow$ The diagonal entries of D are eigenvalues of A, and the columns of P are the corresponding eigenvectors.

Conclusions

Working through ordinary differential equations, it is possible to determine the spectrum of hyponormal matrices by going through matrix operations. Also by using the diagonalization procedure/algorithm, and working through trace and determinants, we can find the characteristic function, eigenvalues and eigenvectors that then enable the evaluation of the spectrum for any symmetric operator matrix. By a similar process, it is also possible to establish the adjoints of most operator matrices. Matrix operations are important because matrices are used to manipulate objects, and its these manipulations that constitute operators. Adjoints relate to denoting a function or quantity by a particular process of transpositions. The term "adjoint" applies in several situations, some with similar formalisms. If A is adjoint to B, then typically there is some formula of the type (Ax, y) = (x, By). The adjoint of an operator A plays the role of a complex conjugate of a complex number, and $adj(A)A = \Delta I$, from which we see $Aadj(A) = \Delta I \Rightarrow \frac{adj(A)}{\Delta} = I$. So in other words, $A^{-1} = \frac{Adj(A)}{\Delta}, \Delta \neq 0$. The eigenvectors of A^n are exactly the same as the eigenvectors of A. The eigenvalues of A^n Are the same as the eigenvalues of A raised to power n, and this rule applies to all integer powers of n, both positive and negative.

Conflicts of interest

Authors declare no conflict of interest.

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