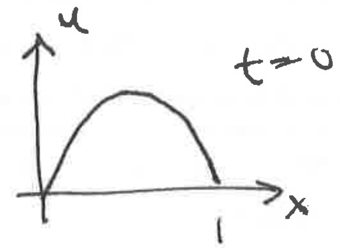
Math 4315 – PDE's

Solve $\quad u_t = u_{xx} \quad 0 < x < 1$

$u_x(t,0) = u_x(t,1) = 0$ insulated Boundaries

$u(0,x) = x - x^2$



Assume

$u = T(t)\,X(x)$ separable Sol^n's

sub $\quad T'X = TX'$

$$\frac{T'}{T} = \frac{X''}{X} = \lambda \quad \text{const (as before)}$$

B.C. $\quad u_x = T(t)\,X'(x)$

$u_x(t,0) = 0 \Rightarrow X'(0) = 0$

$u_x(t,1) = 0 \Rightarrow X'(1) = 0$

so $\dfrac{T'}{T} = \lambda, \quad \dfrac{X''}{X} = \lambda$ or $X'' - \lambda X = 0$

3 cases for λ

(i) $\lambda = \omega^2 > 0$

(ii) $\lambda = 0$

(iii) $\lambda = -\omega^2 < 0$ only case that works with BC.

So $X = c_1 \sin \omega x + c_2 \cos \omega x$

$X' = c_1 \omega \cos \omega x - c_2 \omega \sin \omega x$

$X'(0) = 0 \Rightarrow c_1 = 0$

$X'(1) = -c_2 \omega \sin \omega = 0$

$\Rightarrow \omega = n\pi$ n integer include zero

So $X = c_2 \cos n\pi x$

$$\frac{T'}{T} = -\omega^2 = -(n\pi)^2 \quad \text{Sep}$$

$$T = c_3 e^{-(n\pi)^2 t}$$

$$u = TX = c_3 e^{-(n\pi)^2 t} c_2 \cos n\pi x =$$

$$u = \sum_{n=0}^{\infty} C_2 C_3 e^{-(n\pi)^2 t} \cos n\pi x$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \cos n\pi x$$

Now IC $\quad u(x,0) = x - x^2$

$$\Rightarrow \quad x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{0} \cos n\pi x$$

cosine series

$$a_0 = \frac{2}{1}\int_0^1 (x - x^2)\,dx = 1/3$$

$$a_n = \frac{2}{1}\int_0^1 (x - x^2)\cos n\pi x\,dx$$

$$= -2 \cdot \frac{1 + \cos n\pi}{n^2\pi^2}$$

sol^n $\quad u(x,t) = \frac{1}{6} + \sum_{n=1}^{\infty} -2\left(\frac{1+\cos n\pi}{n^2\pi^2}\right) e^{-(n\pi)^2 t} \cos n\pi x.$