Calculus
Review Test
(w/ solutions)

23+ questions include limits, instantaneous rate of change, integrals, implicit differentiation, maximum/minimum, concavity, and more…

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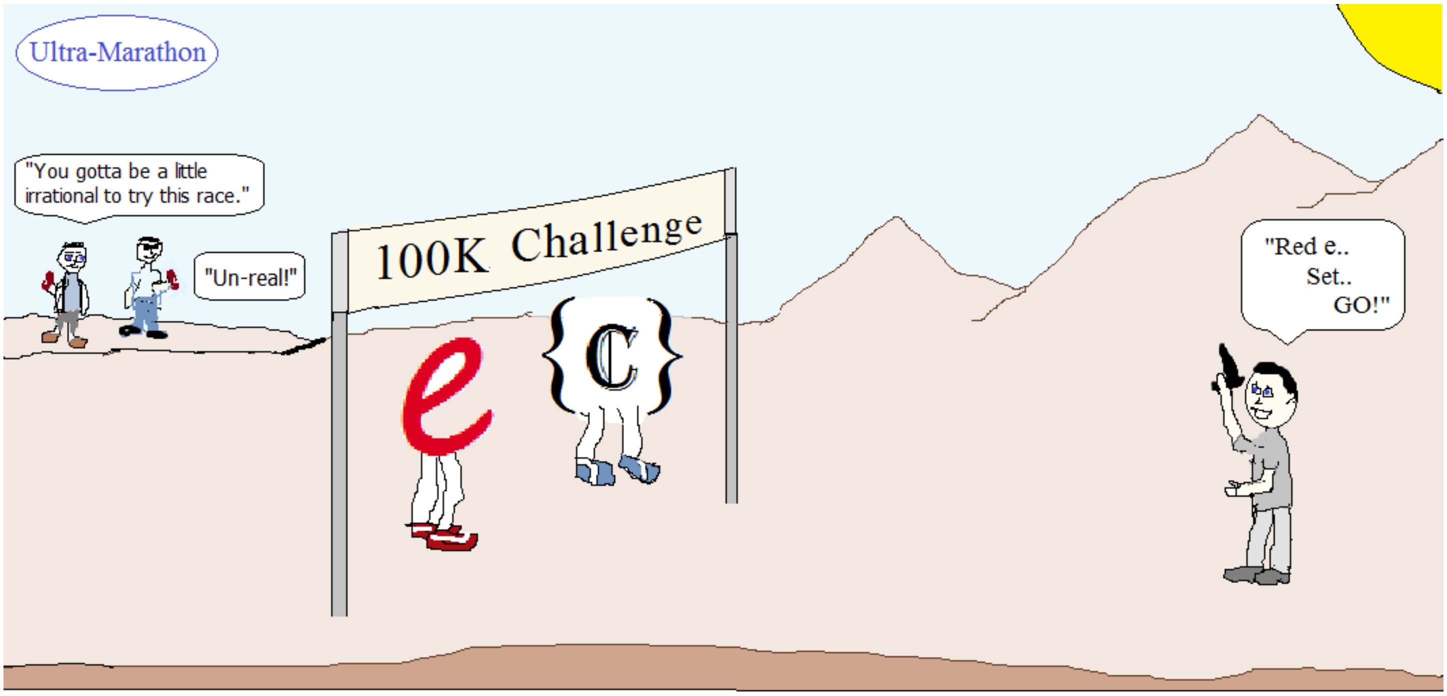

Testing the limits of endurance,
these math figures will run on and on...

LanceAF #87 5-24-13
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Questions

Calculus Review Test

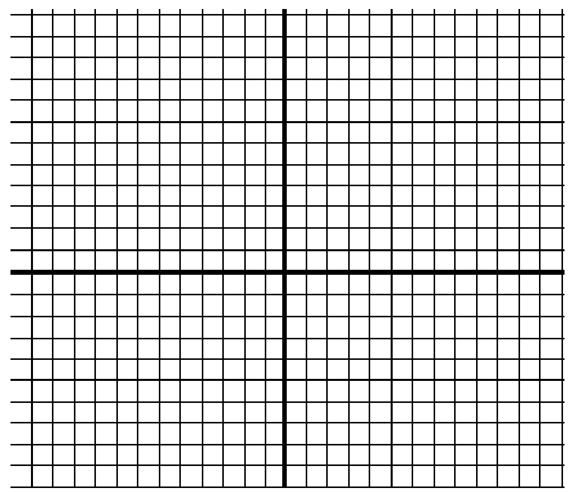
1) $f(x) = x^3 - 2x + 4$

a) Find the average rate of change (AROC) over the interval [-3, 2]

b) Find the instantaneous rate of change (IROC) at the point (1, 3)

2) Find the equation of the line that is tangent to the curve $y = x^2 + 3$ at $x = 2$

Optional: Graph the line and curve, labeling the point of intersection.



3) What is the slope of the *normal* line of $y = 4x^3$ at point $(2, 32)$?

4) Find the $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

Calculus Review Test

5) Evaluate $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

a) <u>without</u> using L'Hospital's Rule

b) using L'Hospital's Rule

6) A cannonball is shot with a trajectory of $h(t) = -5t^2 + 60t$ where t is time (in seconds) and h(t) is height (in feet).

a) What is the *maximum height* of the cannonball?

b) On what interval is the cannonball *going higher*?

c) On what interval is the cannon ball *speeding up*?

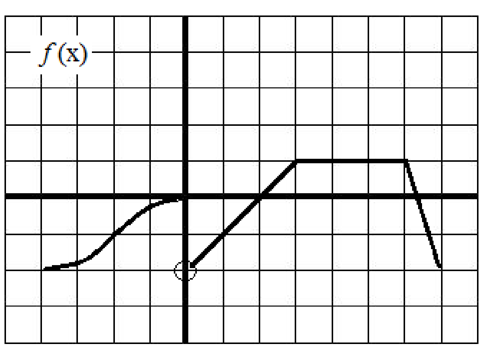
d) How long is the cannonball in the air?

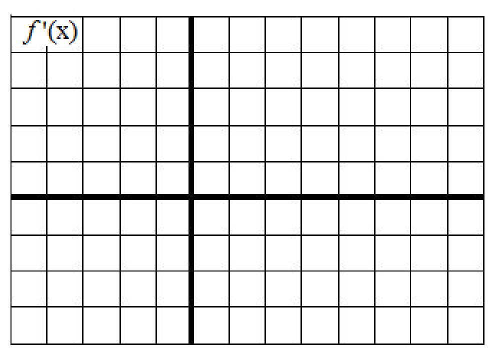
e) What is the *speed* of the cannonball at 10 seconds?

7) Find values for A and B that make the function differentiable at the "breaking point"

$$f(x) = \begin{cases} 2Ax + 5 & \text{if } x < -1 \\ 3x^2 + B & \text{if } x \geq -1 \end{cases}$$

8) Sketch $f'(x)$ on the interval (-4, 7)





Calculus Review Test

9) $f(x) = (4x^3)^3 - 6x$

What is $f'(x)$?

10) $g(x) = \sin^2 x + \cos^2 x - 5$

What is $g'(x)$?

11) Determine if the value is $<$ $>$ or $=$ to zero:

a) $g(1)$

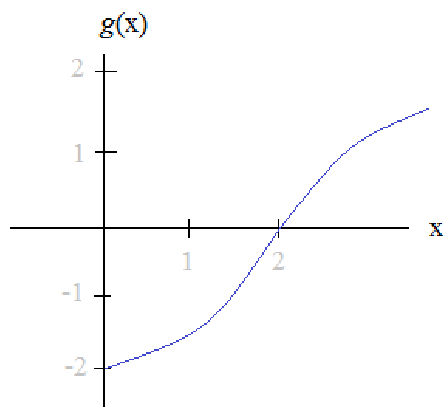
b) $g'(1)$

c) $g''(1)$

d) $g(2)$

e) $g'(2)$

f) $g''(2)$



12) For the function $f(x) = x^2 - 2x - 3$,

a) $f'(x) =$

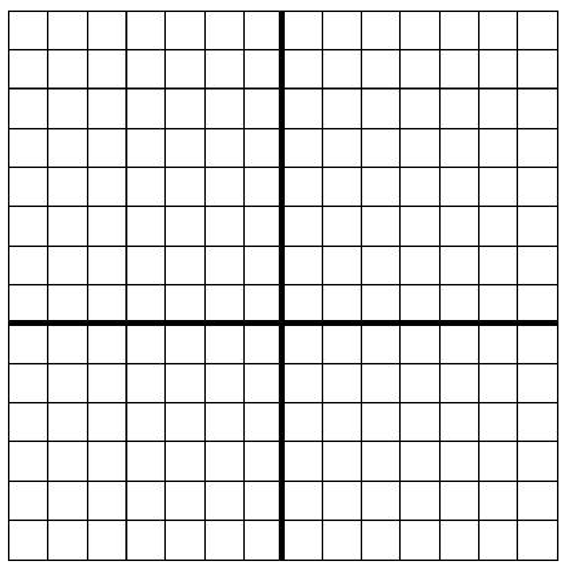
b) $f''(x) =$

c) Identify local extrema:

d) Identify points of inflection:

e) Describe the concavity:

f) Graph the function.

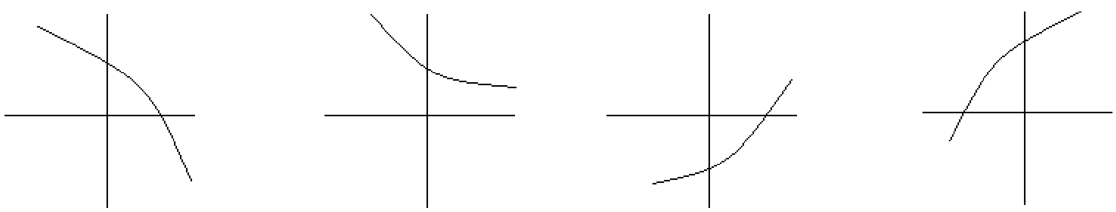


Calculus Review Test

13) If $g(x) = ax + 1$ and $\int_1^2 g(x)\,dx = 7$, what is a?

14) Find the *average* value of $f(x) = x^2 + 3x + 2$ over the interval $[2, 6]$

15) If y is a function of x, $y' > 0$ and $y'' < 0$, which is the graph?



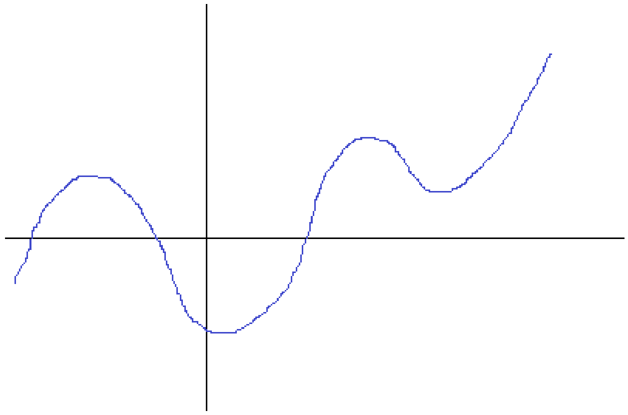
16) Given the function $h(x) = x^5 - 5x^4 + 2x + 3$ Find *all values* where $h(x)$ is concave up.

17) Label any points where

a) $f(x) = 0$

b) $f'(x) = 0$

c) $f''(x) = 0$



18) Find absolute maximum and minimum points of $4x^3 - 15x^2 + 12x + 5$ in the interval $[0, 3]$

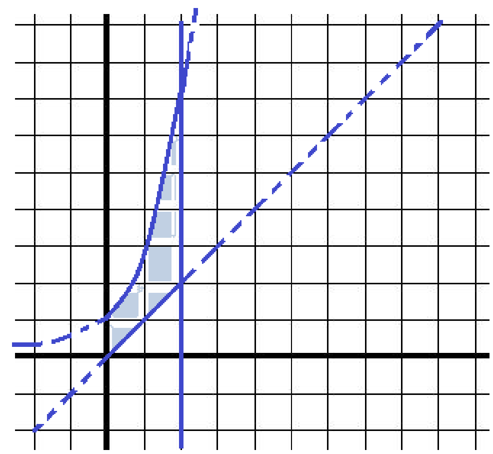
19) Find the extrema of $2x^2 + 5x + 6$ in the interval $[-2, 3]$

20) $\int 2\sin x - 8 + x^3 \; dx$

Calculus Review Test

21) Find the area bordered by:

$y = e^x$
$y = x$
$x = 2$
the y-axis



22) Find the derivative *with respect to x:*

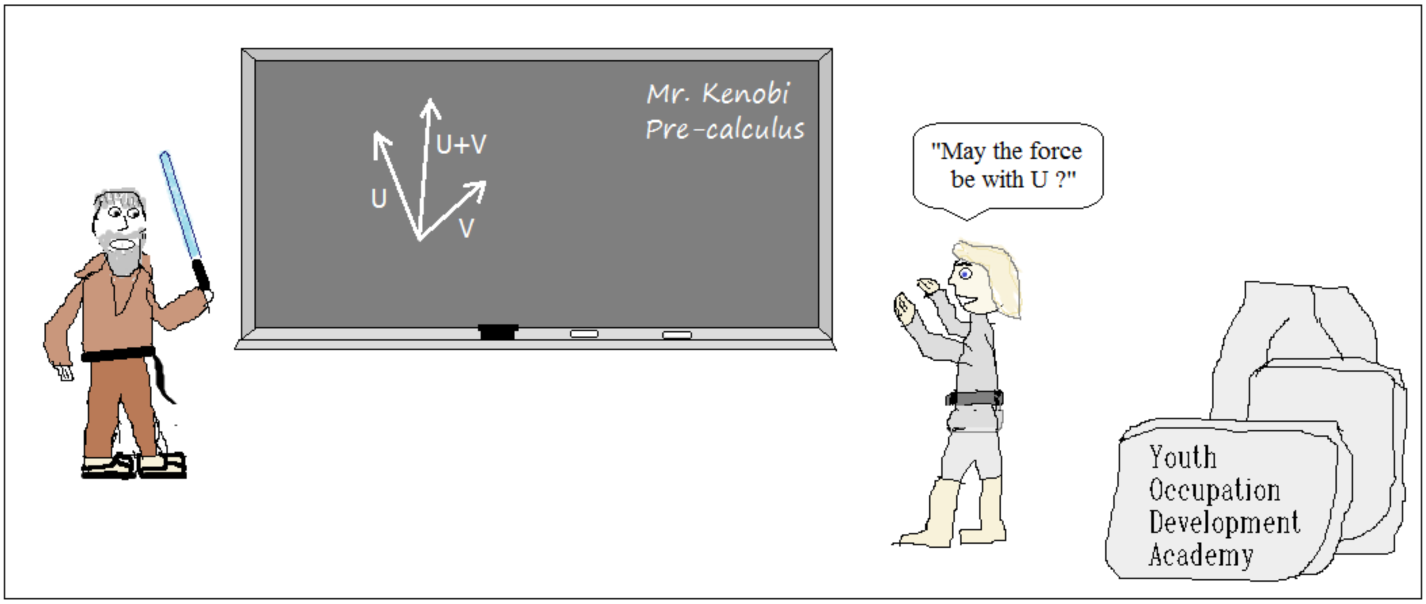
$$x^2 - 2y^3 + 4x = 2$$

(find dy/dx)

23) The *top* of a 20-foot ladder slides down the side of a house at the rate of 6 inches/second. When the *bottom* of the ladder is 16 feet from the house, how fast is the *bottom* of the ladder moving away from the house?

A long time ago,
in a classroom
far, far away...

Math Lessons from the Jedi



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Obi-Wan teaches Luke about resultant vectors and (the) force

Answers

Calculus Review Test

SOLUTIONS

1) $f(x) = x^3 - 2x + 4$

a) Find the average rate of change (AROC) over the interval [-3, 2]

AROC is the "slope" $\dfrac{f(2) - f(-3)}{2 - (-3)} = \dfrac{8 - (-17)}{5} = \boxed{5}$

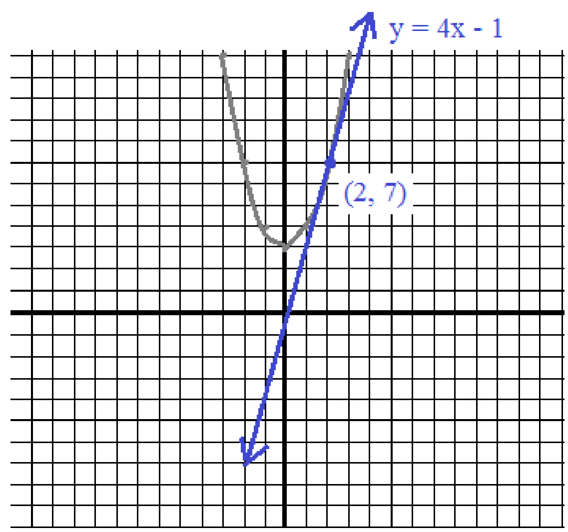
b) Find the instantaneous rate of change (IROC) at the point (1, 3)

"IROC is the first derivative" $f'(x) = 3x^2 - 2$

IROC @ (1, 3) $f'(1) = 3(1)^2 - 2 = \boxed{1}$

2) Find the equation of the line that is tangent to the curve $y = x^2 + 3$ at $x = 2$

Optional: Graph the line and curve, labeling the point of intersection.



To determine the equation of a line, we need the slope and a point:

Slope: $y' = 2x$ (1st derivative)

at $x = 2$, the slope is 4

Point: $y = x^2 + 3$

at $x = 2$, $y = 7$ (2, 7)

equation of the line tangent to the curve: $\boxed{y - 7 = 4(x - 2)}$

or $y = 4x - 1$

3) What is the slope of the *normal* line of $y = 4x^3$ at point (2, 32) ?

A normal line is perpendicular to the tangent. $y' = 12x^2$ at $x = 2$, $y' = 48$

Since the slope of the tangent is 48, then the slope of the normal line is $\boxed{-\dfrac{1}{48}}$

4) Find the $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$

step 1: substitution

$\dfrac{1 - \cos(0)}{(0)^2} = \dfrac{0}{0}$ inconclusive

step 2: L'Hospital's Rule

$\dfrac{0 - (-\sin x)}{2x} = \dfrac{\sin x}{2x}$

(substitution) $\dfrac{\sin(0)}{2(0)} = \dfrac{0}{0}$

(repeat) $\dfrac{\cos x}{2}$ (substitution)

$\dfrac{\cos(0)}{2} = \boxed{\dfrac{1}{2}}$

SOLUTIONS

Calculus Review Test

5) Evaluate $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

a) without using L'Hospital's Rule

(since direct substitution results in 0/0, we try the conjugate)

$$\frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \frac{1}{(\sqrt{x} + 3)}$$

$$\lim_{x \to 9} \frac{1}{(\sqrt{x} + 3)} = \boxed{\frac{1}{6}}$$

b) using L'Hospital's Rule

derivative of numerator: $\frac{1}{2} x^{\frac{-1}{2}} - 0 = \frac{1}{2\sqrt{x}}$

derivative of denominator: 1

$$\lim_{x \to 9} \frac{1}{2\sqrt{x}} = \boxed{\frac{1}{6}}$$

6) A cannonball is shot with a trajectory of $h(t) = -5t^2 + 60t$ where t is time (in seconds) and h(t) is height (in feet).

$h'(t) = -10t + 60$

a) What is the *maximum height* of the cannonball?

180 feet

$h'(t) = 0$ when $t = 6$

max height is $-5(6)^2 + 60(6) = 180$

b) On what interval is the cannonball *going higher*?

(0, 6)

when $h'(t) > 0$, the cannonball height is increasing.. This occurs between 0 and 6

c) On what interval is the cannon ball *speeding up*?

(6, 12)

$h''(t) = -10$ so, cannonball is decelerating.. and, the cannonball goes downward from 6 sec. to 12 sec

d) How long is the cannonball in the air?

12 seconds

find where $h(t) = 0$

$0 = -5t^2 + 60t$

$= -5t(t^2 - 12)$

$t = 0, 12$

e) What is the *speed* of the cannonball at 10 seconds?

40 feet/second

at 10 seconds, $h'(10) = -10(10) + 60 = -40$

so, the speed is $|-40| =$ 40 feet/second

7) Find values for A and B that make the function differentiable at the "breaking point"

$$f(x) = \begin{cases} 2Ax + 5 & \text{if } x < -1 \\ 3x^2 + B & \text{if } x \geq -1 \end{cases}$$

Function must be differentiable AND continuous at x = -1

(continuous)

$2Ax + 5 = 3x^2 + B$ @ x = -1

$-2A + 5 = 3 + B$

$2A + B = 2$

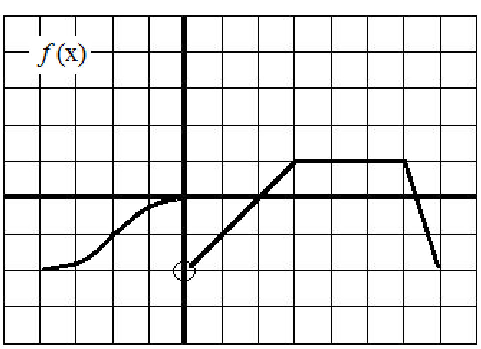
(differentiable)

$2A + 0 = 6x + 0$ @x = -1

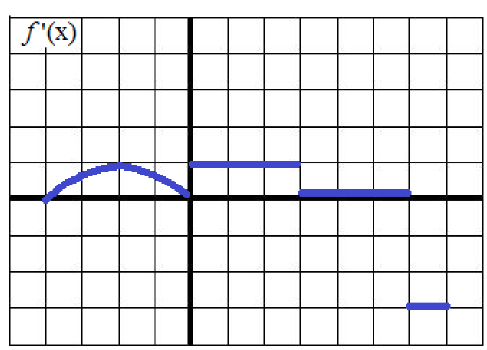
$2A = -6$

$\boxed{A = -3 \text{ then, } B = 8}$

8) Sketch $f'(x)$ on the interval (-4, 7)



$f'(x)$ represents the instanteous rates of change of $f(x)$



Calculus Review Test

SOLUTIONS

9) $f(x) = (4x^3)^3 - 6x$

What is $f'(x)$?

chain rule: $3(4x^3)^2\ (12x^2) - 6$

$3(16x^6)(12x^2) - 6$

$576x^8 - 6$

10) $g(x) = \sin^2 x + \cos^2 x - 5$

What is $g'(x)$?

shortcut: $\sin^2 + \cos^2 = 1$ (trig identity)

$g(x) = 1 - 5 = -4$

$g'(x) = 0$

long way: $g'(x) = 2\sin x^1(\cos x) + 2\cos x^1(-\sin x) - 0$

$= 2\sin x\cos x - 2\cos x\sin x$

$= 0$

11) Determine if the value is $<$ $>$ or $=$ to zero:

a) $g(1)$ < 0 (output below x axis)

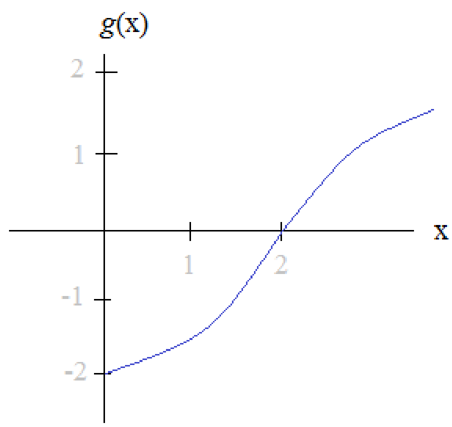
b) $g'(1)$ > 0 (slope positive)

c) $g''(1)$ > 0 (concave up)

d) $g(2)$ $= 0$

e) $g'(2)$ > 0

f) $g''(2)$ $= 0$ (point of inflection)



12) For the function $f(x) = x^2 - 2x - 3$,

a) $f'(x) =$ $2x - 2$

b) $f''(x) =$ 2

c) Identify local extrema: $x = 1$ (minimum) or $(1, -4)$

f'(x) = 0 at x = 1

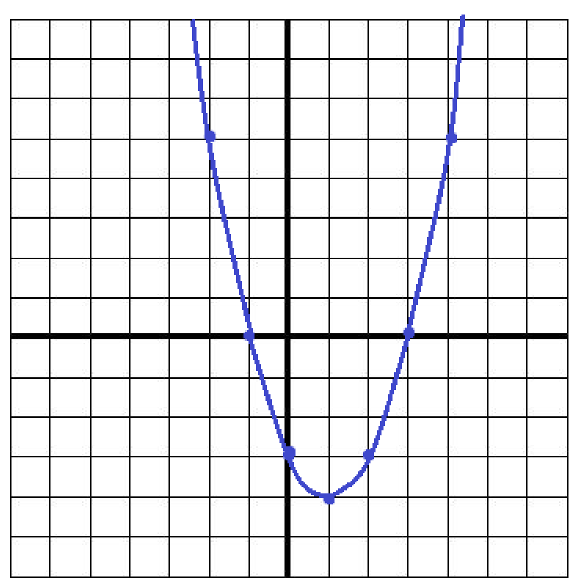
d) Identify points of inflection: NONE

f''(x) is never equal to zero

e) Describe the concavity: since $f''(x) > 0$ at all points, the function is *concave up*

f) Graph the function.

y-intercept: (0, -3)
x-intercepts: (3, 0) (-1, 0)
axis of symmetry: x = 1
vertex: (1, -4)



SOLUTIONS

Calculus Review Test

13) If $g(x) = ax + 1$ and $\int_1^2 g(x)\,dx = 7$, what is a?

$$7 = \frac{ax^2}{2} + x \Big|_1^2 = \frac{a(2)^2}{2} + (2) - \left(\frac{a(1)^2}{2} + (1) \right)$$

$$= 2a + 2 - a/2 - 1 = \frac{3a}{2} + 1 \qquad \boxed{a = 4}$$

check: $\int_1^2 4x + 1 = 2x^2 + x \Big|_1^2$

$8 + 2 - (2 + 1) = 7$ ✓

14) Find the *average* value of $f(x) = x^2 + 3x + 2$ over the interval [2, 6]

Find the total value under the curve: $\int_2^6 x^2 + 3x + 2\, dx = \frac{x^3}{3} + \frac{3x^2}{2} + 2x \Big|_2^6 = 72 + 54 + 12 - (8/3 + 6 + 4)$

$= 138 - 38/3 = 125\frac{1}{3}$

or 376/3

Then, find average value: $\frac{\text{total value}}{\text{interval}} = \frac{376/3}{4} = \frac{376}{12} = \boxed{31\,\frac{1}{3}}$

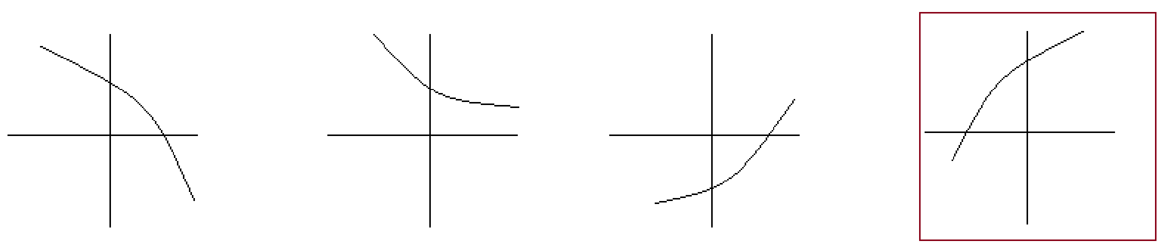
Quick check for "reasonableness":

$f(2) = 12 \quad f(4) = 30 \quad f(6) = 56$

31.33 seems to be a potential average ✓

15) If y is a function of x, $y' > 0$ and $y'' < 0$, which is the graph?

since $y' > 0$, function must be increasing (slope)
and, since $y'' < 0$, function must be concave down



16) Given the function $h(x) = x^5 - 5x^4 + 2x + 3$ Find *all values* where $h(x)$ is concave up.

$h'(x) = 5x^4 - 20x^3 + 2$

$h''(x) = 20x^3 - 60x^2$ →

Determine where the second derivative > 0

$20x^3 - 60x^2 = 0$

$20x^2(x - 3) = 0$

$x = 0, 3$ (points of inflection)

$x < 0$: $h''(x)$ is negative

$0 < x < 3$: $h''(x)$ is negative

$x > 3$: $h''(x)$ is positive

therefore, $h(x)$ is concave up where $x > 3$

SOLUTIONS

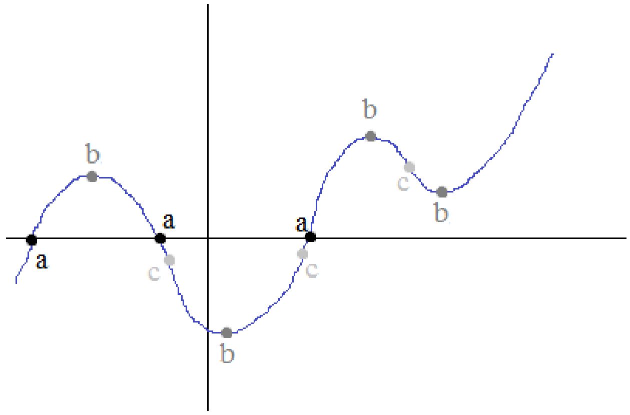
Calculus Review Test

17) Label any points where

a) $f(x) = 0$

b) $f'(x) = 0$

c) $f''(x) = 0$



a: function touches x-axis
b: tangent is horizontal line
c: points of inflection

18) Find absolute maximum and minimum points of $4x^3 - 15x^2 + 12x + 5$ in the interval [0, 3]

Find first derivative and set equal to zero...

$f' = 12x^2 - 30x + 12$

if $f' = 0$, $12x^2 - 30x + 12 = 0$ divide both sides by 6

$2x^2 - 5x + 2 = 0$ factor

$(2x - 1)(x - 2) = 0$

$x = 1/2$ and 2

**Both are in the interval [0, 3]

If x = 2:

$4(2)^3 - 15(2)^2 + 12(2) + 5 = 1$

If x = 1/2:

$4(1/2)^3 - 15(1/2)^2 + 12(1/2) + 5 = 7.75$

Test points:		
0: $f(0) = 5$	(2, 1)	$(1/2, 7\frac{3}{4})$
1: $f(1) = 6$	absolute minimum	*relative* maximum
3: $f(3) = 14$		

In the interval [0, 3], the absolute max is (3, 14)... and, the absolute min is (2, 1)...

19) Find the extrema of $2x^2 + 5x + 6$ in the interval [-2, 3]

first derivative: $4x + 5$

set equal to zero to find the max/min: $4x + 5 = 0$

$x = -5/4$

at x = -2, the output is 4
at x = 3 , the output is 39

so, (3, 39) is maximum

2nd derivative: 4

since it is greater than zero, concave up ----> minimum

(-5/4, 23/8) is the minimum..

SOLUTIONS

Calculus Review Test

20) $\int 2\sin x - 8 + x^3 \, dx$

$\int 2\sin x \, dx - \int 8 \, dx + \int x^3 \, dx$

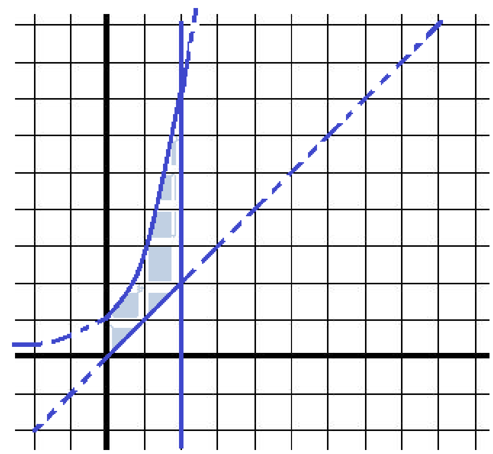
$-2\cos x - 8x + \frac{x^4}{4} + C$

21) Find the area bordered by:

$y = e^x$
$y = x$
$x = 2$
the y-axis

The shaded area consists of the area under the log function MINUS the right triangle (i.e. the area under the line y = x)

(use definite integral to find area)



$y = e^x$

x	y
-1	.37
0	1
1	2.72
2	7.39

(approx.)

$A_{log} = \int_0^2 e^x \, dx = e^2 - e^0$

$= 7.39 - 1$

$A_{tri} = \frac{1}{2}(2)(2) = 2$

Total Area is approx. 4.39 sq. units

22) Find the derivative *with respect to x:*

$x^2 - 2y^3 + 4x = 2$

(find dy/dx)

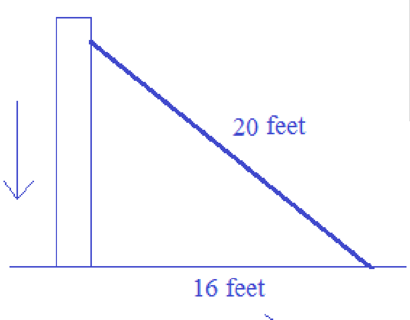
$2x - 6y^2\frac{dy}{dx} + 4 = 0$

$\frac{dy}{dx} = \frac{-2x - 4}{-6y^2}$

$\frac{dy}{dx} = \frac{x + 2}{3y^2}$

23) The *top* of a 20-foot ladder slides down the side of a house at the rate of 6 inches/second. When the *bottom* of the ladder is 16 feet from the house, how fast is the *bottom* of the ladder moving away from the house?

Step 1: Diagram and relevant formulas.



$\frac{dy}{dt} = \frac{-.5 \text{ ft}}{\text{per sec}}$

$\frac{dx}{dt}$

pythagorean theorem:

$x^2 + y^2 = \text{hypotenuse}^2$

Step 2: Create the equation that we need to solve.

$x^2 + y^2 = 20^2$

Find change of distance from house with respect to time...... $\frac{dx}{dt}$

The *top* of a 20-foot ladder slides down the side of a house at the rate of 6 inches/second. When the *bottom* of the ladder is 16 feet from the house, how fast is the *bottom* of the ladder moving away from the house?

Step 3: Solve the equation

.5 ft/sec = 6 inches/sec

$$x^2 + y^2 = 20^2$$

Use implicit differentiation to find the change with respect to time (t).

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

since $x^2 + y^2 = 20^2$

when $x = 16$, $y = 12$

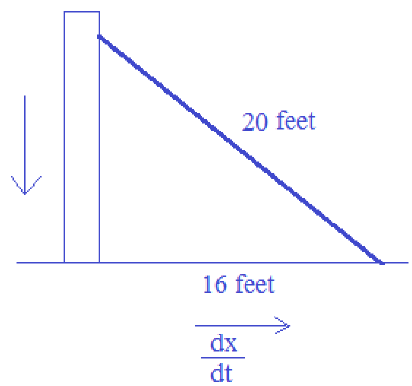
and, $\frac{dy}{dt} = -.5$

$$2(16)\frac{dx}{dt} + 2(12)(-.5) = 0$$

$$32\frac{dx}{dt} - 12 = 0$$

$$\frac{dx}{dt} = \frac{3}{8} \text{ feet/second}$$

4.5 inches/second



Step 4: Check the answer →

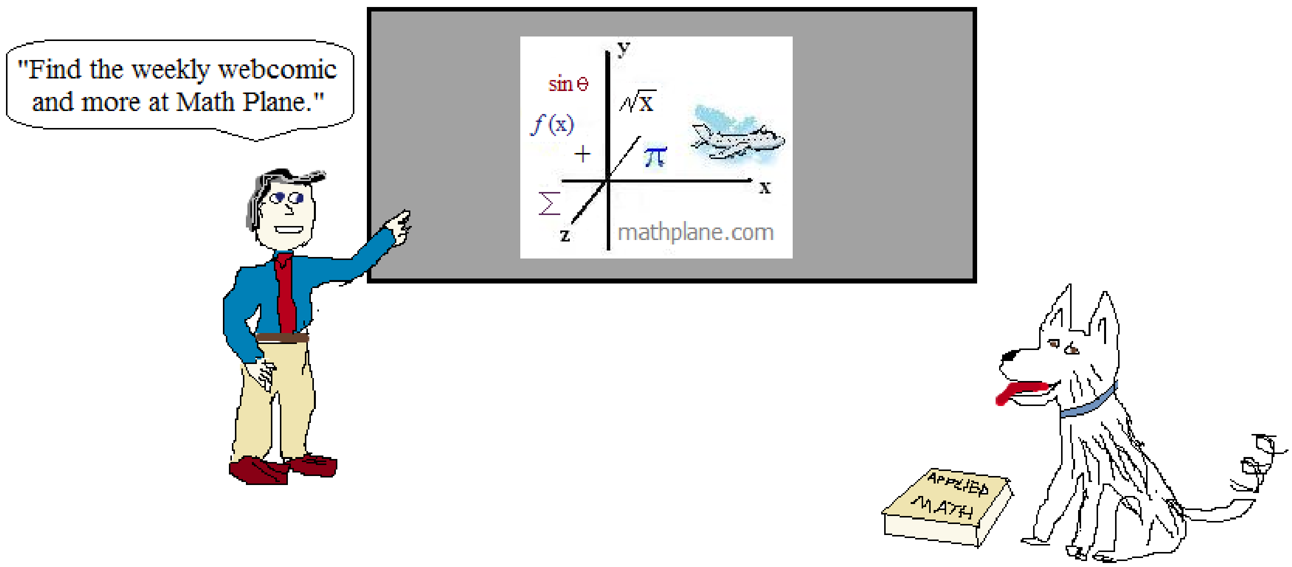
elapsed time t	distance x	height y	average rate of x over 4 second intervals
0	12	16	.57 feet/sec
4	14.28	14	.43 feet/sec
8	16	12	.33 feet/sec
12	17.32	10	.25 feet/sec
16	18.33	8	

.375 feet/sec

Thanks for checking out this Calculus Review. (Hope it helped!)

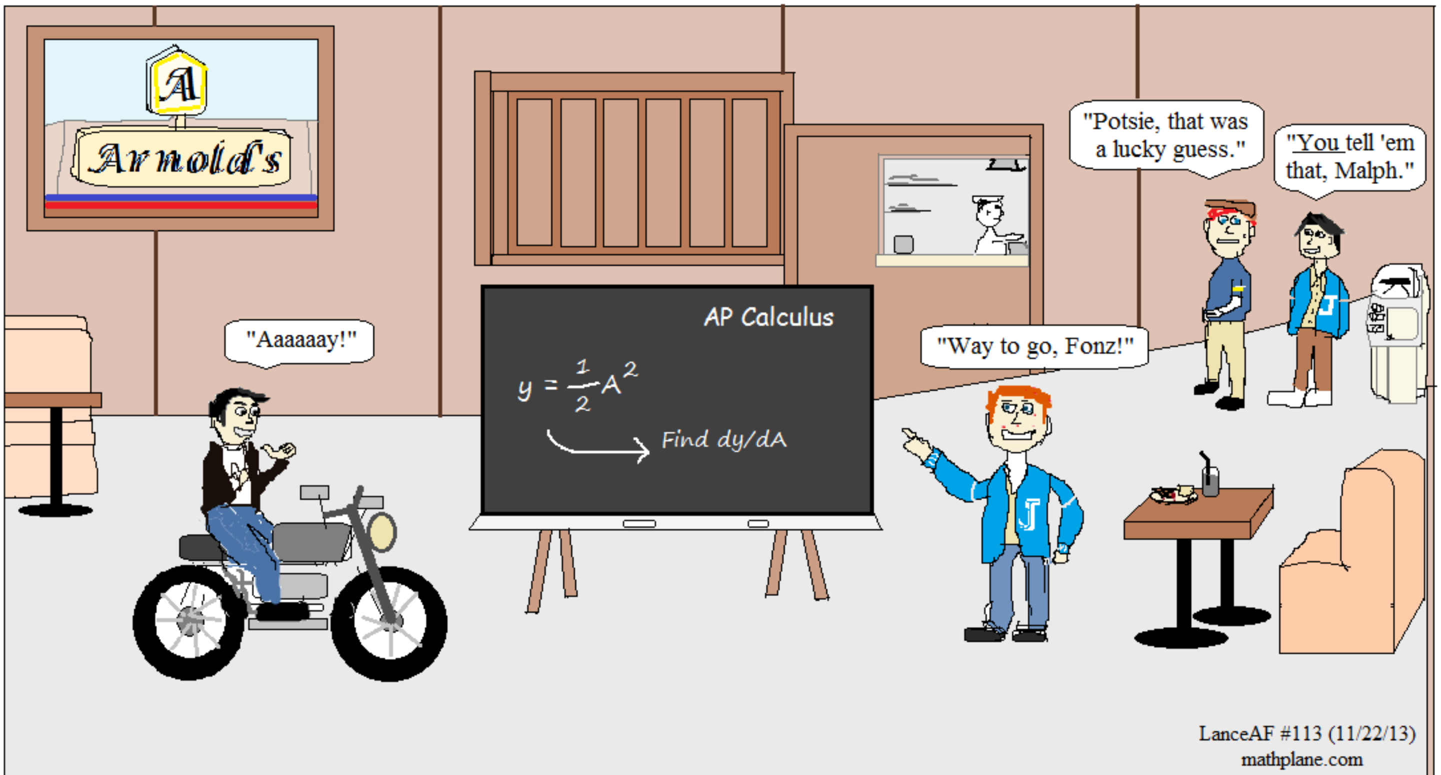
If you have any questions, suggestions, or feedback, let me know.

Cheers,
Lance



We appreciate your support.

(All proceeds go to the site and treats for my dog, Oscar!)



Happy Days

Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

Also, at Facebook, Google+, and Pinterest