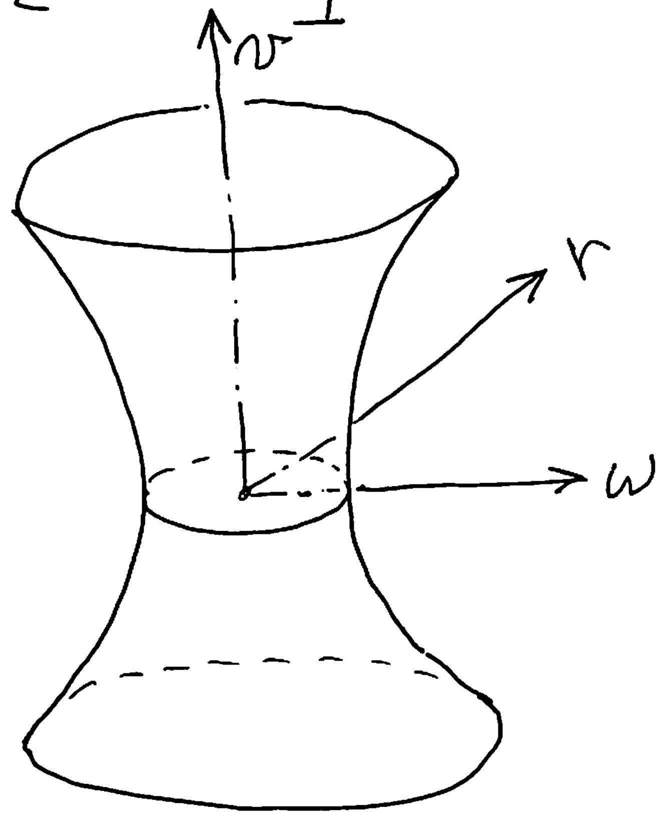
Appendix A: DeSitter Cartography

- DeSitter space is the surface of a hyperboloid in five dimensional Minkowski space:

$$v^2 - w^2 - x^2 - y^2 - z^2 = -1$$

- Trivialize θ and ϕ:

$$v^2 - w^2 - r^2 = -1$$



- --

- "deSitter space is the surface of a hyperboloid" is like saying "the surface of the earth is a sphere."
- Maps are still very useful guides in finding one's way around. It is the same with deSitter space.
- Our purpose:

  Catalogue deSitter line elements (metric forms)
  Relate them to each other.
  Relate them all to u, v, and w.
  In Appendix B, connect them to FS and spinors.

The Cartography Catalogue

- There will be ten entries.

  (1 – 3): FRW cosmology (k =1, 0, -1)

  (4 – 6): The conformally flat versions of the above FRW metrics.

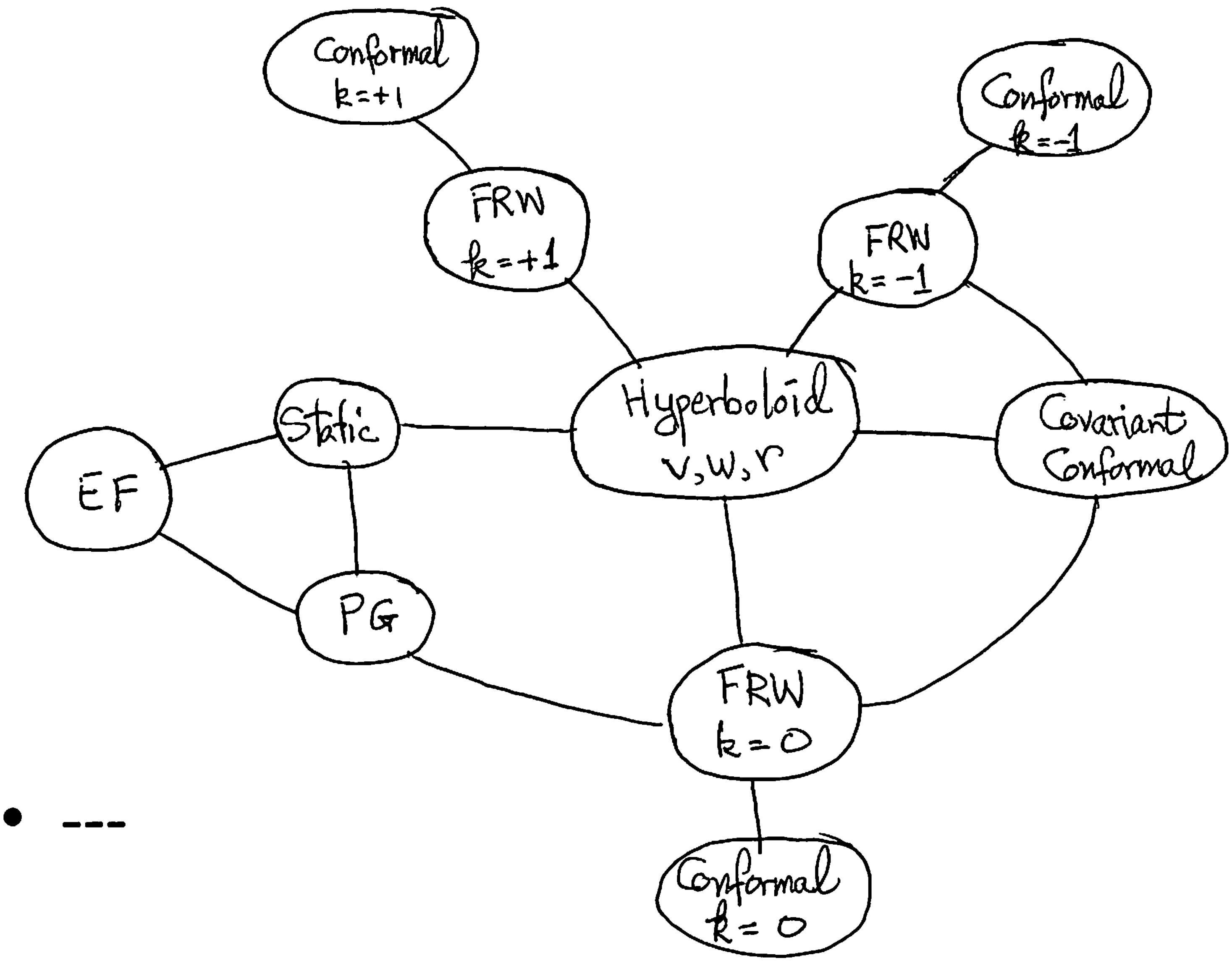
  (7): “Covariant conformal” metric (not yet introduced).

  (8): Static deSitter space (not yet introduced).

  (9): Painleve-Gullstrand.

  (10): Eddington-Finkelstein (not yet introduced).

The Web of Cartographies



- ---

"Hyperspherical" Coordinates

- Three ways of introducing the Minkowskian version of spherical coordinates:

- The first way:

$$V = \sinh t$$
$$W = \cosh t \cos \chi$$
$$r = \cosh t \sin \chi$$

$$\boxed{\begin{aligned} d\Omega &\equiv d\theta^2 + \sin^2\theta\, d\varphi^2 \\ \sin\chi &\equiv \rho \\ d\rho &= \cos\chi\, d\chi \end{aligned}}$$

$$ds^2 = dt^2 - \cosh^2 t\,(d\chi^2 + \sin^2\chi\, d\Omega) = dt^2 - \cosh^2 t\left(\frac{d\rho^2}{1-\rho^2} + \rho^2 d\Omega\right)$$

- This gives k = +1 FRW

$$a(t) = \cosh t \qquad \boxed{\left(\frac{\dot a}{a}\right)^2 + \frac{1}{a^2} = 1}$$

- The second way:

$$\boxed{\begin{aligned} \rho &= \sinh \chi \\ d\rho &= \cosh \chi \, d\chi \end{aligned}}$$

$$\begin{aligned} W &= \cosh t \\ V &= \sinh t \cosh \chi \\ r &= \sinh t \sinh \chi \end{aligned}$$

$$ds^2 = dt^2 - \sinh^2 t \left(d\chi^2 + \sinh^2 \chi \, d\Omega \right) = dt^2 - \sinh^2 t \left(\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega \right)$$

$$a(t) = \sinh t \qquad \boxed{\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = 1}$$

- This gives k = -1 FRW cosmology

•

- The third way:

$$r = \sin \chi$$
$$w = \cos \chi \cosh t$$
$$v = \cos \chi \sinh t$$

$$ds^2 = \cos^2 \chi \, dt^2 - \left(d\chi^2 + \sin^2 \chi \, d\Omega\right) = \left(1-r^2\right) dt^2 - \frac{dr^2}{(1-r^2)} - r^2 d\Omega$$

- This gives static deSitter space!!

- Where is k = 0 FRW?

"Light-cone" coordinates:

$$w + v = e^{t}$$
$$w - v = e^{-t} - \rho^2 e^{t}$$

$$\boxed{\begin{aligned} w^2 - v^2 &= 1 - \rho^2 e^{2t} \\ \therefore\ r &= \rho e^{t} \end{aligned}}$$

[Note: $dw^2 - dv^2 = -dt^2 + e^{2t} d\rho^2 - dr^2$]

$$ds^2 = dt^2 \pm e^{2t} d\rho^2 - r^2 d\Omega$$

or

- -- $$ds^2 = dt^2 - e^{2t}\left(d\rho^2 + \rho^2 d\Omega\right)$$

The three "conformally flat" versions

- k = 0:

$$dt = e^{t}\, d\eta \qquad ds^2 = \frac{1}{\eta^2}\left(d\eta^2 - d\rho^2 - \rho^2 d\Omega\right)$$

- k = + 1:

$$dt = \cosh t \, d\eta \qquad ds^2 = \frac{1}{\eta^2}\left(d\eta^2 - \frac{d\rho^2}{1-\rho^2} - \rho^2 d\Omega\right)$$

- k = - 1:

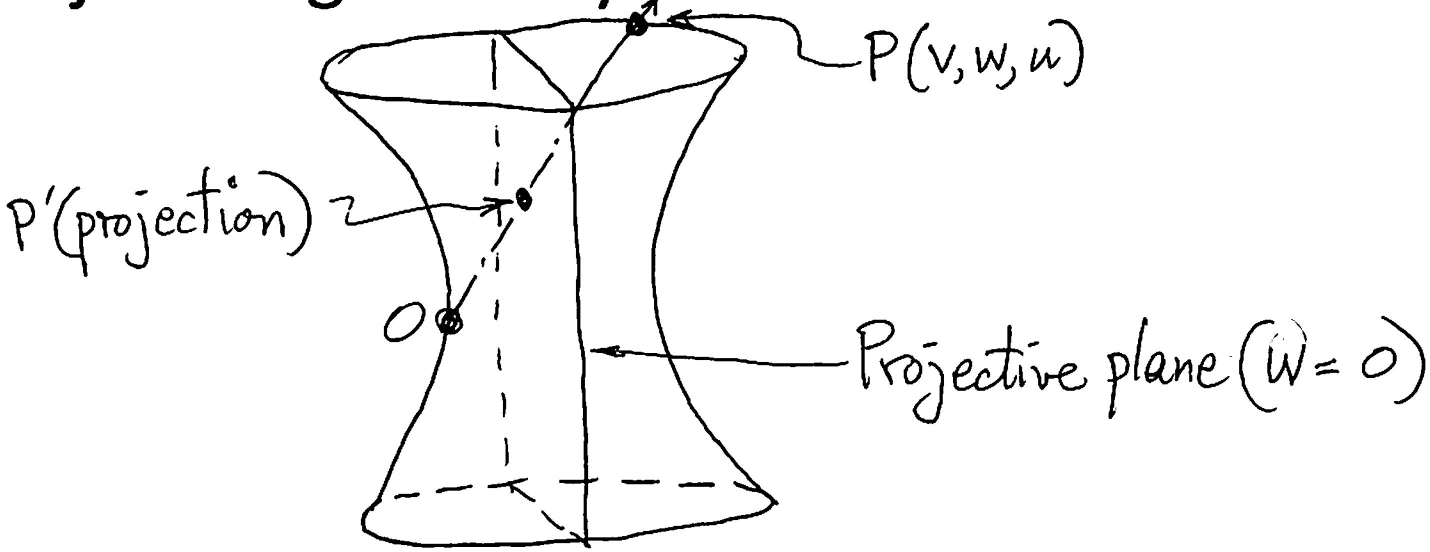
$$dt = \sinh t \, d\eta \qquad ds^2 = \frac{1}{\eta^2}\left(d\eta^2 - \frac{d\rho^2}{1+\rho^2} - \rho^2 d\Omega\right)$$

- Useful for cosmological perturbations; also for Penrose diagrams.

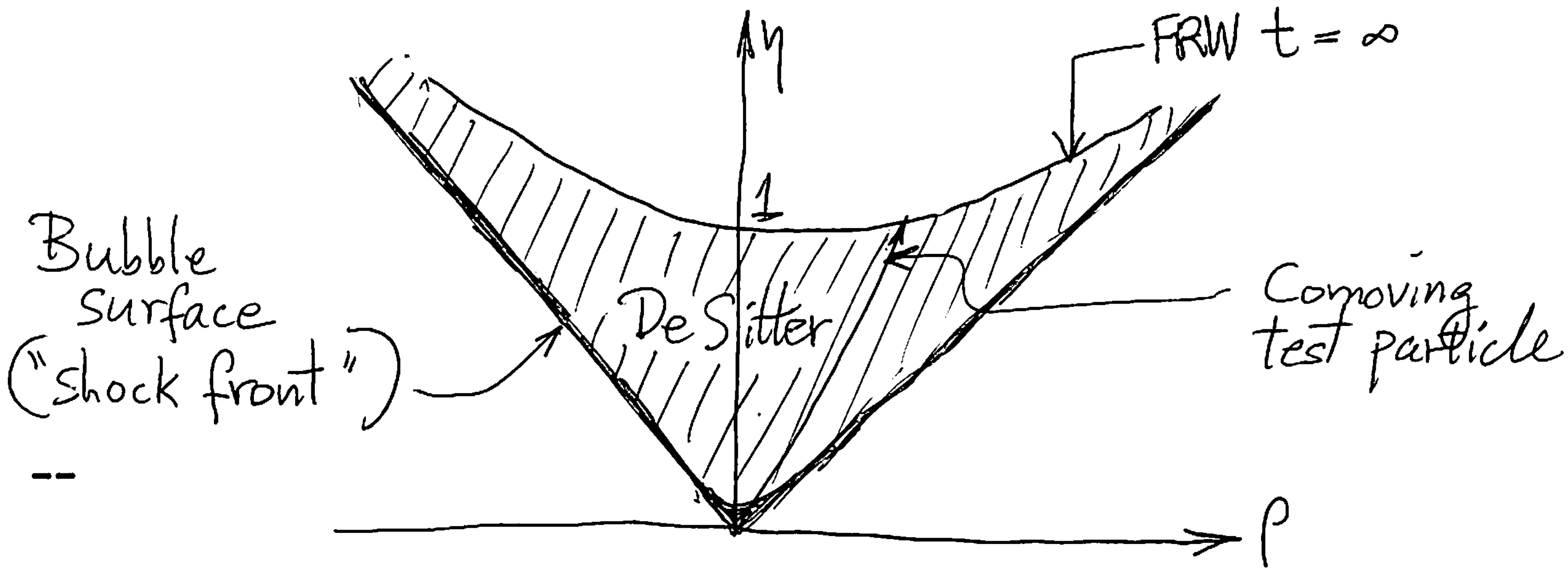
Covariant conformal coordinates

- Not in the books. But they ought to be.
  Cognoscenti know and use them.
  See Lasenby and Doran: astro-ph / 0411579
- The line element: $$ds^2 = \frac{4(d\eta^2 - d\rho^2 - \rho^2 d\Omega)}{(1 - \eta^2 + \rho^2)^2}$$
- Relate it to k = -1 FRW via
$$\eta = \tanh\frac{t}{2}\cosh\chi \qquad \rho = \tanh\frac{t}{2}\sinh\chi$$
- Solve for v,w,r in terms of η and ρ:
$$\eta = \frac{v}{w+1} \qquad \rho = \frac{r}{w+1}$$
- It is projective geometry.

- The projective geometry construction:



- This metric is ideal for "bubble cosmology"



- --

- Connection of covariant-conformal coordinates to static deSitter coordinates:

$$\eta = \frac{\sinh t \cos\chi}{(1+\cosh t\cos\chi)} \qquad \rho = \frac{\sin\chi}{(1+\cosh t\cos\chi)}$$

$$\left[ds^2 = \cos^2\chi\, dt^2 - d\chi^2 - \sin^2\chi\, d\Omega\right]$$

- Connection of covariant-conformal coordinates to k = 0 FRW coordinates:

$$e^{\tau} = \frac{(1+\eta)^2 - \rho^2}{(1-\eta^2+\rho^2)} \qquad \sigma = \frac{2\rho}{(1+\eta)^2 - \rho^2}$$

$$\left[ds^2 = d\tau^2 - e^{2\tau}(d\sigma^2 + \sigma^2 d\Omega)\right]$$

- --

The Painleve-Gullstrand Metric

- Start from flat FRW and change variables as before:

$$ds^2 = d\tau^2 - e^{2\tau}(d\sigma^2 + \sigma^2 d\Omega)$$

$$= d\tau^2 - (d\rho - \rho\, d\tau)^2 - \rho^2 d\Omega$$

$$\boxed{\rho = e^{\tau}\sigma}$$

- Since

$$W + V = e^{\tau}$$
$$W - V = e^{-\tau}(1-\rho^2)$$

$$W^2 - V^2 = 1 - e^{-2\tau}\rho^2 = 1 - r^2$$

- The PG radial variable r is the same r as for the deSitter hyperboloid.

The Eddington Finkelstein Metric

- Definition of the metric:

$$ds^2 = dt^2 - dr^2 - r^2(dt - dr)^2 - r^2 d\Omega$$

- It is closely connected to PG. Start with general stationary PG metric.

$$ds^2 = d\tau^2 - (dr - v(r)\,d\tau)^2 - r^2 d\Omega$$

- Shift the origin of time in a nonsingular, r-dependent way as follows:

$$t = \tau + \int_0^r \frac{dr\, v(r)}{(1 + v(r))}$$

- The EF metric emerges, with the same velocity function as for PG:

$$ds^2 = dt^2 - dr^2 - v^2(r)(dt - dr)^2 - r^2 d\Omega$$

- --

- A digression:
- This EF metric is a special case of Kerr-Schild metrics, defined as follows:

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega - (l_\mu dx^\mu)^2$$

- The vector l_μ is null; for our case

$$l_\mu = (|v(r)|, v(r), 0, 0)$$

- The KS form is useful in describing rotating black holes.

- Back to the catalog. Use the PG result to determine the relation of EF variables to the hyperboloid variables:

First; $t = \tau + r - \ln(1+r)$

- From here one gets

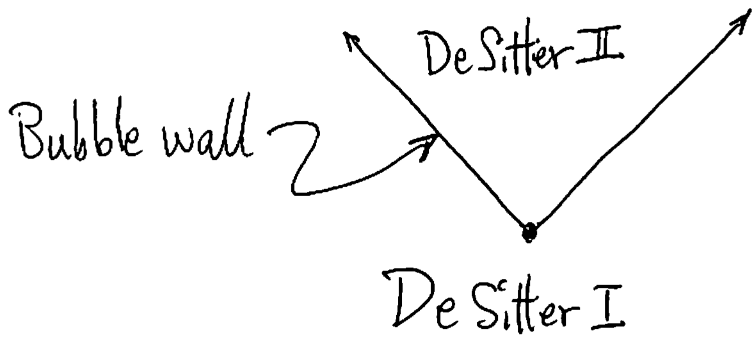
$$w + v = e^{\tau} = (1+r)e^{(t-r)}$$
$$w - v = e^{-\tau}(1-r^2) = (1-r)e^{-(t-r)}$$

- The bottom line is quite simple:

$$w + v = (1+r)e^{(t-r)}$$
$$w - v = (1-r)e^{-(t-r)}$$

- --

- EF coordinates are useful for describing cosmological bubble formation:



$$\mu_1^4 > \mu_2^4$$

- It is easy to determine the structure of the light-cone boundary "shock front" from the Einstein equations.

  The shock front contains massless quanta.

  The energy in the shock accounts for the dark-energy deficit in the bulk interior of the bubble.

--