VANDERBILT UNIVERSITY School of Engineering

Discrete Structures

CS 2212

(Fall 2020)

19 – Counting

Reminder and Recap …

Recap:

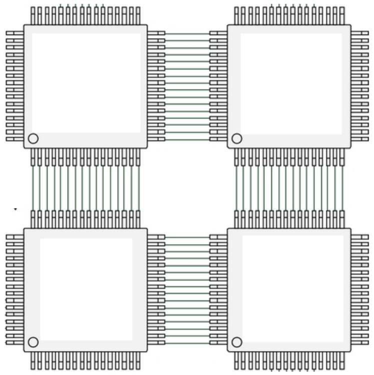
- **Last time**
  - We completed Chapter 8 (Integers)
- **Today**
  - Chapter 9 (Counting)
  - Permutations and Combinations (without repetitions)

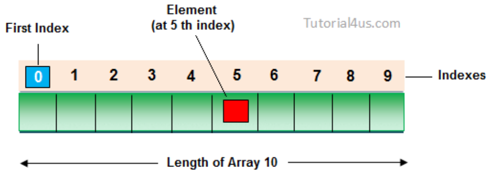
Counting

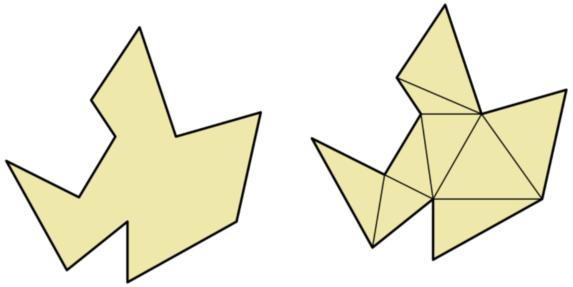
Counting is related to a branch of discrete math called **Combinatorics**.

Apart from beautiful theory, Counting is an important mathematical tool to analyze many practical problems.

- How many possibilities/ways are there …?
- How many possibilities are there under certain constraints …?
- Chances of a possible outcome …







Revisiting the Product Rule

The **Product rule:**

$$|A \times B| = |A|\,|B|$$

This rule can be extended as in

$$|A \times B \times C| = |A|\,|B|\,|C|$$

Example: How many strings of length 4 are there over the alphabet { a, b, c }?

Answer: $|3|^4 = 3 * 3 * 3 * 3 = 81$

Revisiting the Product Rule

The **Product rule** provides a way to count **sequences**

Example: Restaurant "complete dinner" special

Choose from 2 appetizers { *Wings, Nachos* }

Choose from 3 meals { *Fish, Chicken, Steak* }

Choose from 2 desserts { *Mud Pie, Cheesecake* }

Question: How many different ways can you order a complete dinner special?

Answer: 2 * 3 * 2 = 12

Revisiting the Sum Rule

The **Sum rule:**

$$|\mathbf{A} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}|$$

where, A and B are disjoint, that is, $A \cap B$ = empty.

This rule can be extended as in

$$|\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}| = |\mathbf{A}| + |\mathbf{B}| + |\mathbf{C}|$$

Revisiting the Sum Rule

The **sum rule** is applied when there are multiple choices but only **one** selection is made.

Example: Restaurant "Drink" special

The following drinks are on special for $1.99:

2 hot drinks { *coffee, hot chocolate* }

3 cold drinks { *juice, milk, soda* }

Question: How many different possible choices for a drink are there for a customer?

Answer: $2 + 3 = 5$

Revisiting the Sum Rule

Consider the following definitions for sets of valid password characters:

- Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }
- Special characters = { *, &, $, # }

Question: How many password of length 6 can be created if passwords can be special characters, digits, or letters?

Answer: 10 + 26 + 4 = 40 choices for each position.

There are 6 positions.

So total choices are: 40^6

The Generalized Product Rule

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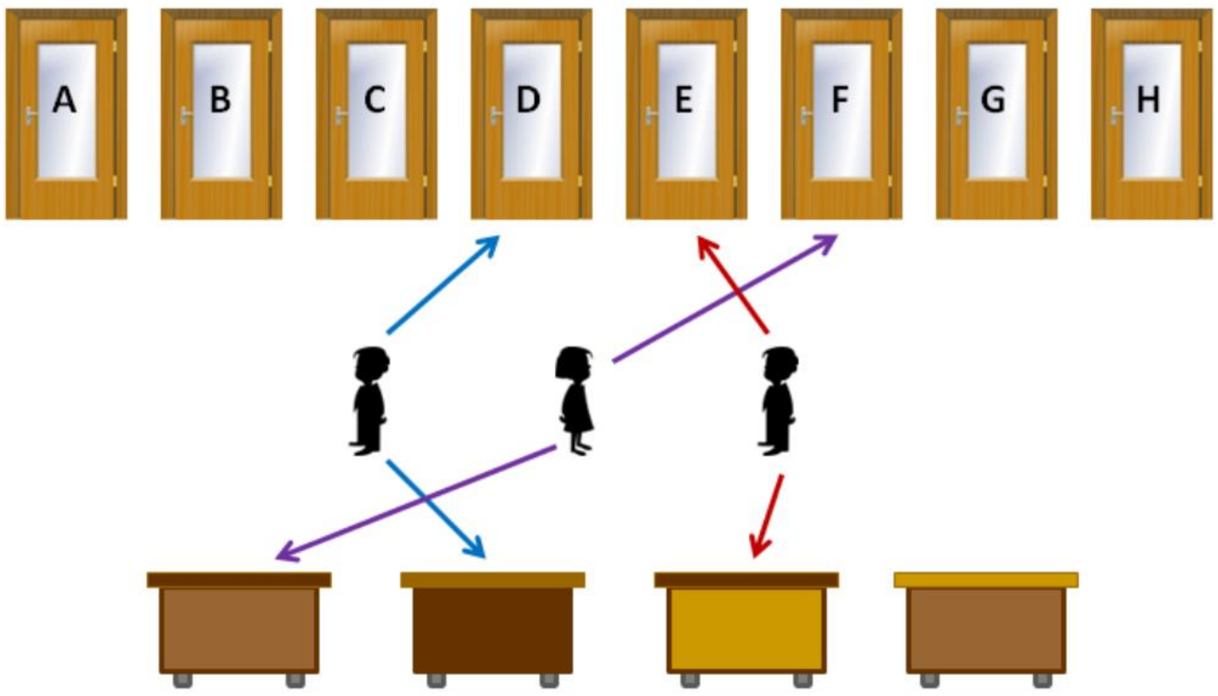
- Special characters = { *, &, $, # }

Question: How many password of length 6 can be created if passwords can be special characters, digits, or letters with **no repeated characters**?

Answer: 10 + 26 + 4 = 40 choices for first position. 39 for second, etc. = 40 * 39 * 38 * 37 * 36 * 35

The Generalized Product Rule

In selecting an item from a set, if the number of choices at each step does not depend on previous choices made, then the number of items in the set is the product of the number of choices in each step.



Three employees rent an office space with **8 offices** and **4 desks**. Each person can select an office and a desk. The selection is done in the order that the participants joined the company with the founder going first. How many ways are there for the selection to be done?

The Generalized Product Rule

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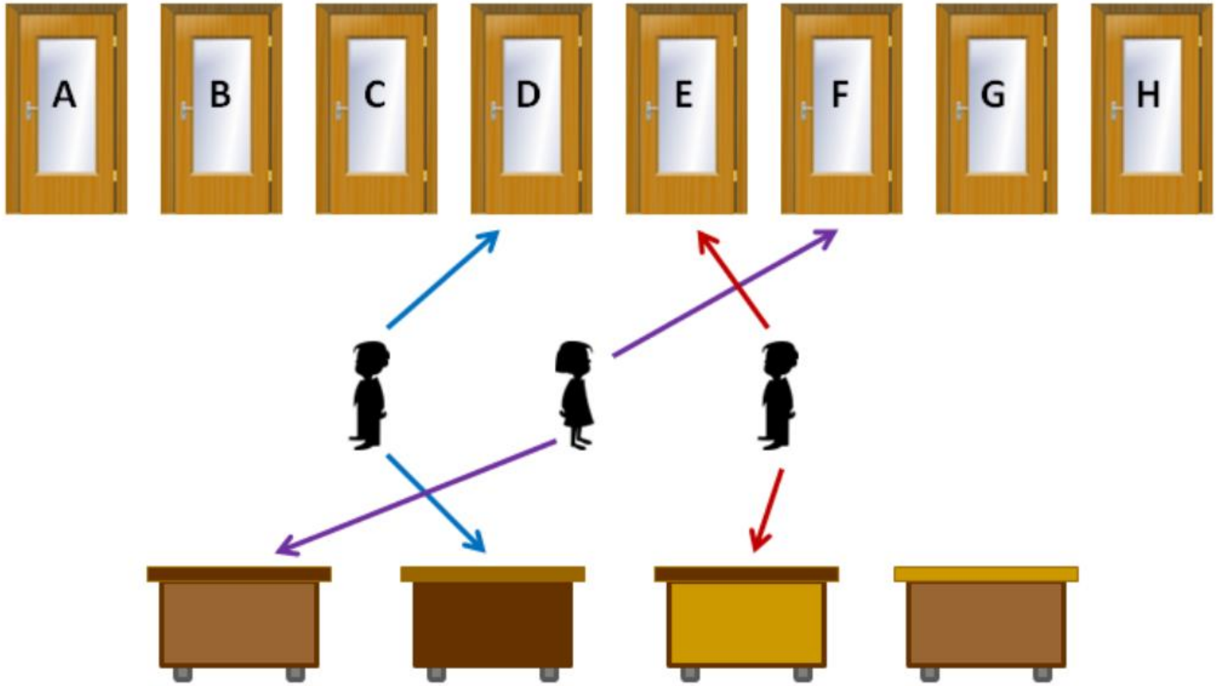

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$$(8 \cdot 4) \cdot (7 \cdot 3) \cdot (6 \cdot 2) = 8064$$

Counting: Using Set Properties

- We can also compare the cardinality of two sets by properties of functions:
  - Recall **surjection** from set A to set B, then **|A| ≥ |B|**
  - **injection** from set A to set B, then **|A| ≤ |B|**
  - **bijection** between sets A and B, then **|A| = |B|**.

- Recall that the **bijection rule** says that if there is a bijection from one set to another then the two sets have the **same** cardinality.

Permutations and Combinations – Motivation

Consider we have a set of four objects.

$$\text{Set} = \{\, \alpha, \beta, \gamma, \delta \,\}$$

In how many ways, can we select **three** objects (no repetitions).

Order **is** important

Which objects are selected, and what is their order?

(**Sequence** of three objects)

Order **is not** important.

All that matters is, which objects are selected?

(**Subset** of three objects)

Set = { α , β , γ , δ }

Spot 1	Spot 2	Spot 3
α	β	δ
α	β	γ
α	γ	β
α	γ	δ
α	δ	γ
α	δ	β
β	α	γ
β	α	δ
β	γ	α
β	γ	δ
β	δ	α
β	δ	γ

Spot 1	Spot 2	Spot 3
γ	α	δ
γ	α	β
γ	β	α
γ	β	δ
γ	δ	α
γ	δ	β
δ	α	β
δ	α	γ
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{ α , β , γ }

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δ	γ	β

Permutations

{ α , β , γ }
{ α , β , δ }
{ β , δ , γ }
{α , δ , γ }

Combinations

Permutations

An ***r*-permutation** is a *sequence of r items with no repetitions*, all taken from the same set.

Permutation Formula: The number of permutations of length *r* chosen from an *n*-element set (where $r \leq n$) is:

$$P(n, r) = \frac{n!}{(n-r)!}; \qquad 0! = 1$$

Example:

$$P(7, 4) = \frac{7!}{(7-4)!} = \frac{7 * 6 * 5 * 4 * 3 * 2 * 1}{3 * 2 * 1}$$

$= 7 * 6 * 5 * 4$

$= 840$

Counting Permutations

Example: Let's say 10 people are competing in a race. We have 3 unique awards (gold, silver, bronze). How many ways can we award the medals among the ten contestants?

$$P(10, 3) = \frac{10!}{(10-3)!} = \frac{10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1}{7 * 6 * 5 * 4 * 3 * 2 * 1}$$

$$= 10 * 9 * 8 = 720$$

Counting Permutations

Example: Counting permutations of passwords.
Each character in a password is either a digit [0-9] or lowercase letter [a-z]. How many valid passwords are there with the given restriction(s)?

Length is 11.
No character repeats.
P(36,11)

Length is 10.
No character repeats.
Starts with **w3**
P(34,8)

Length is 9.
No character repeats.
Must contain **a**
P(9,1) × P(35,8)

Length is 8.
No character repeats.
Must contain **a, b, c**
P(8,3) × P(33,5)

Combinations

Combinations are similar to permutations, but **the order does not matter** in terms of the results.

Example: Let's say 10 people are competing in a race. I have *3 generic medals* for awards. How many ways can we award the medals among the 10 contestants?

How to think about it: If I give a medals to Jim, Mary and Jane, that's the same as giving medals to Jane, Mary and Jim; the same as Mary, Jane, Jim. So we have a lot more "duplicates" we need to remove.

Combinations

An ***r*-combination** is a *subset of of r items* taken from the same set.

The number of ways of selecting an *r*-subset from a set of size *n* is:

$$C(n,k) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Example:

$$C(10,3) = \binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10 \, * 9 \, * 8}{3 \, * 2 \, * 1}$$

= 5 * 3 * 8 = 120

Combinations

Examples: 14 students have volunteered for a committee. 8 of them are seniors and 6 of them are juniors.

How many ways are there to select a committee with 3 seniors and 2 juniors ?

$$\binom{8}{3}\binom{6}{2}$$

Suppose the committee must have five students (either juniors or seniors) and that one of the five must be selected as chair. How many ways are there to make the selection?

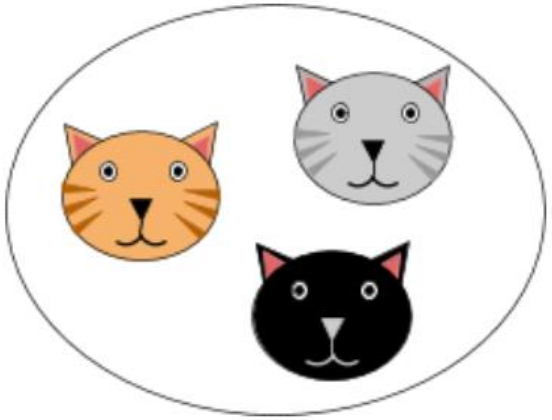
$$14\binom{13}{4}$$

Combinations vs Permutations

A shelter has 20 cats.

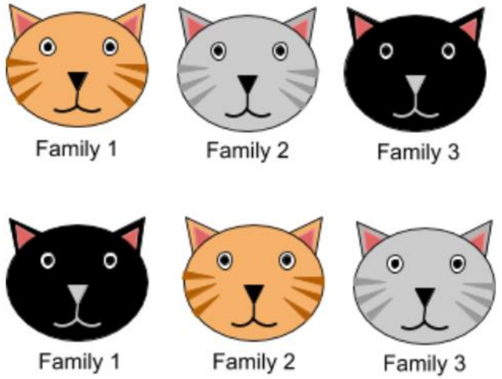
A family selects 3 cats. How many ways are there in which a family can select three cats?

$$\binom{20}{3}$$



Now, there are three families, and each selects 1 cat. How many ways are there for the families to make selections?

$$P(20,3)$$



Combinations vs Permutations

A country has two political parties, the Ds and the Rs. Suppose the senate consists of 56 members of Ds and 44 members of Rs.

How many ways are there to select a committee of 4 senate members with the same number of Ds and Rs?

$$\binom{56}{2}\binom{44}{2}$$

Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

$$P(56,2)\ P(44,2)$$

Counting by Complement

Count the number of elements in a set S that have a certain property by counting the total number of elements in S and subtracting the number of elements in S that do not have the property.

Total number of **all** possible ways	−	Number of possible ways **not satisfying** the criteria	=	Number of possible ways **satisfying** the criteria

Counting by Complement

Example: Counting permutations of passwords.

Each character in a password is either a digit [0-9] or lowercase letter [a-z]. How many passwords of length 9 are there such that no character is repeated and the password must contain a ?

Total ways of selecting a password of length 9.

P(36,9)

Ways of selecting a password of length 9 with no “a”.

P(35,9)

Number of ways to select password of length 9 containing “a”

P(36,9) – P(35,9)

(Previously, we solved it by P(9,1) × P(35,8)).

Counting by Complement

Example:

A shop has 7 different shirts and 8 different jeans. How many ways are there to select 2 items so that at least one jeans is chosen?

Total ways of selecting 2 items from 15 different items.

$$\binom{15}{2}$$

Number of ways in which 2 shirts will be selected.

$$\binom{7}{2}$$

Number of ways in which at least 1 jean will be selected:

$$\binom{15}{2} - \binom{7}{2}$$

Counting – Permutations and Combinations

Consider a set (collection) of ***n*** objects.

Set = { α , β , γ , . . . , δ }

What is the total number of **Sequences** of length ***r***?

[____ ____ ____ ____ ____]

- Order is important
- Permutation problem
- Elements in the sequence are not repeated (so far).

$$P(n,k) = \frac{n!}{(n-r)!}$$

Counting – Permutations and Combinations

Consider a set (collection) of ***n*** objects.

Set = { α , β , γ , . . . , δ }

What is the total number of **Subsets** of length ***r***?

{____ ____ ____ ____ ____}

- Order is not important
- Combination problem
- Elements in the subset are not repeated (so far).

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$