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The MacDowell-Mansouri Extension: Addendum

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I. Gauss-Bonnet Numerology: Some Derivations

We begin by deriving the formula for the "topological density n(t) (actually a constant) for deSitter space. The problem is essentially keeping track of numerical factors.

Begin with the appropriate Lagrangian

$$
\begin{aligned}
L = {} & C_{EH}\left(\frac{M_{pl}^2}{8\pi}\right)\int d^3\xi\, a(t)^3\left[3\left(\frac{\ddot a}{a}\right)+3\left(\frac{\dot a}{a}\right)^2\right] \\
& - C_{cc}\left(\frac{3M_{pl}^2 H^2}{8\pi}\right)\int d^3\xi\, a(t)^3 \\
& - C_{GB}\left(\frac{M_{pl}^2}{8\pi H^2}\right)\int d^3\xi\, a(t)^3\left[3\left(\frac{\ddot a}{a}\right)\left(\frac{\dot a}{a}\right)^2\right]
\end{aligned}
$$

We have assumed in the above that the Riemann tensor, up to a normalization factor, has the form appropriate to k = 0 FRW cosmology, with a(t) the FRW scale factor. In Petrov form, it is given as follows:

$$
R_{\mu\nu}^{\ \ \alpha\beta} =
\begin{array}{c}
t\xi_x \\ t\xi_y \\ t\xi_z \\ \xi_y\xi_z \\ \xi_z\xi_x \\ \xi_x\xi_y \\ \\
\end{array}
\begin{array}{c}
\left(\begin{array}{ccc|ccc}
\left(\frac{\ddot a}{a}\right) & 0 & 0 & & & \\
0 & \left(\frac{\ddot a}{a}\right) & 0 & & \bigcirc & \\
0 & 0 & \left(\frac{\ddot a}{a}\right) & & & \\
\hline
 & & & \left(\frac{\dot a}{a}\right)^2 & 0 & 0 \\
 & \bigcirc & & 0 & \left(\frac{\dot a}{a}\right)^2 & 0 \\
 & & & 0 & 0 & \left(\frac{\dot a}{a}\right)^2
\end{array}\right) \\
\begin{array}{cccccc}
t\xi_x & t\xi_y & t\xi_z & \xi_y\xi_z & \xi_z\xi_x & \xi_x\xi_y
\end{array}
\end{array}
$$

The variables $\vec{\xi}$ in the expression for the Lagrangian are the comoving FRW coordinates. The integral of those coordinates over a comoving volume is independent of time. Therefore the time-dependent volume V(t) scales as the cube of a(t) .

The structure of the Gauss-Bonnet term, up to its normalization, follows from its definition

$$L_{GB} \sim \frac{M_{pl}^2}{H^2} \int d^3\xi \sqrt{g} \left[R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} \epsilon_{\alpha\beta\gamma\delta} \right]$$

The problem is to make sure that the normalizations of the three terms in the Lagrangian are correct, namely that

$$C_{EH} = C_{cc} = C_{GB} = 1$$

We do this in three steps. The first is to recognize that the cosmological-constant term can be written as

$$L_{cc} = -\mu^4 \int d^3\xi \sqrt{g} = -\mu^4 \int d^3\xi \, a(t)^3$$

Here μ is the dark-energy density, related to the Hubble expansion rate (for pure deSitter space) by the FRW equation.

$$H^2 = \frac{8\pi}{3M_{pl}^2} \mu^4$$

Therefore

$$C_{cc} = 1$$

Next, normalization of the Einstein term can be determined by applying the variational principle to this Lagrangian. First one makes an integration by parts on the Einstein term, along with discarding temporarily the Gauss-Bonnet term, because it does not affect the equations of motion. The integration by parts gives

$$L_{eff} \propto \left[C_{EH}(\ddot{a}a^2 + \dot{a}^2 a) - C_{cc} a^3 \right] \Rightarrow -\left[C_{EH} \dot{a}^2 a + C_{cc} a^3 \right]$$

The equation of motion becomes

$$\frac{d}{dt}\left[2C_{EH} \dot{a} a \right] = C_{EH} \dot{a}^2 + 3 C_{cc} a^2$$

This equation is satisfied for the solution $a(t) = e^{Ht}$, provided $C_{EH} = C_{cc} = 1$.

To determine the Gauss-Bonnet coefficient, we demand, as explained in the text, that the Lagrangian, as given by the first equation in this addendum, must vanish when the equations of motion are satisfied. This leads to the equation

$$2C_{EH} - C_{cc} - C_{GB} = 0$$

Therefore $C_{GB} = 1$, and

$$L_{GB} = -\left(\frac{M_{pl}^2}{8\pi H^2}\right)\int d^3\xi \frac{d}{dt}(\dot{a})^3 \equiv -2\pi \frac{dN}{dt}$$

Therefore the density is given by

$$N(t) = \frac{M_{pl}^2}{16\pi^2 H^2}\left[H^3 V(t)\right] \Rightarrow n = \frac{HM_{pl}^2}{16\pi^2} = \frac{4\pi HM_{pl}^2}{(4\pi)^3}$$

With the input numbers,

$M_{pl}^{-1} = 1.6 \times 10^{-33}$

H = deSitter Hubble length = 1.5×10^{28} cm.

the output density is

$$n \approx \left[\frac{4\pi HM_{pl}^2}{\Lambda_{QCD}^3}\right] \cdot \left(\frac{\Lambda_{QCD}}{4\pi}\right)^3 \approx \frac{1}{3}\left(\frac{\Lambda_{QCD}}{4\pi}\right)^3 \approx \left(\frac{\Lambda_{QCD}}{20}\right)^3 \approx 10\,\text{MeV}$$

We here choose $\Lambda_{QCD} = 200$ MeV. Evidently the result is well on the infrared side of the QCD scale. This occurs in large part because of the numerical coeffiecient $(4\pi)^3$, something not anticipated by Zeldovich in 1967.

II. Cosmology: Derivation of the Numbers

From the preceding formulae and the argument in the notes, it is straightforward to write down the expression for the critical temperature. The missing ingredient is the Stefan-Boltzmann expression for the temperature. It is

$$\rho_{rad} = \frac{\pi^2 g^*}{30} T^4 \quad \Rightarrow \quad H^2 = \frac{8\pi}{3M_{pl}^2}\left(\frac{\pi^2 g^*}{30} T^4\right) = \left(\frac{4\pi^3 g^*}{45}\right)\left(\frac{T^2}{M_{pl}}\right)^2$$

Here g^* expresses the effective number of degrees of freedom of the primordial soup at temperature T. Therefore the equation for the critical temperature T_c becomes

$$m_c \equiv M_{pl}^3 = \frac{M_{pl}^2}{16\pi^2 H^2}\left(\frac{4\pi^3 g^*}{45}\right)^{3/2} \frac{T_c^6}{M_{pl}^3}$$

$$T_c = \left[4\pi H M_{pl}^2\right]^{1/3} \left(\frac{45}{4\pi^3 g^*}\right)^{1/4}$$

At the appropriate value of T, one finds from page 65 of Kolb and Turner a value $g^* = 15$. This leads finally to

$$T_c \approx \frac{\Lambda_{QCD}}{4} \approx 50 \text{ MeV}$$

III. Neutron Stars and the Like: More Numbers

The Riemann tensor for the general stationary Painleve-Gullstrand metric is remarkably simple (see Fischer and Visser, arXiv 0205139, as well as Hamilton, arXiv 0411060,for good overviews):

$$R^{\alpha\beta}_{\ \ \mu\nu} = \begin{array}{c} tr \\ t\theta \\ t\varphi \\ \theta\varphi \\ r\varphi \\ r\theta \end{array} \left(\begin{array}{ccc|ccc} \frac{u''}{2} & 0 & 0 & & & \\ 0 & \frac{u'}{2r} & 0 & & 0 & \\ 0 & 0 & \frac{u'}{2r} & & & \\ \hline & & & \frac{u}{r^2} & 0 & 0 \\ & 0 & & 0 & \frac{u'}{2r} & 0 \\ & & & 0 & 0 & \frac{u'}{2r} \end{array}\right)$$

Here we have defined

$$u \equiv v^2 \qquad u' \equiv \frac{du}{dr}$$

We therefore can read off the form of the Gauss-Bonnet term:

$$L_{GB} \sim \left[\frac{uu''}{2r^2} + 2\left(\frac{u'}{2r}\right)^2 \right] = \frac{1}{2r^2}\frac{d}{dr}(uu')$$

We now consider a volume comoving with the Painleve-Gullstrand flow with dimensions

$$\Delta V = (\Delta r) \times (r\,\Delta\theta) \times (r\,\Delta\varphi) = r^2\,\Delta\theta\,\Delta\varphi\,\Delta r$$

As the box falls in toward the source, the distance between the floor and the ceiling increases, and is given by

$$\Delta r = v(r)\,\Delta t$$

Here Δt is time independent, because the trajectory of the floor is obtained from that of the ceiling by a time translation. Therefore

$$\Delta V(t) = r^2(t)\,v(t)\;\Delta\theta\;\Delta\varphi\;\Delta t$$

Consequently, the GB $\Delta N(t)$ within this box is

$$S_{GB} = 2\pi\,\Delta N(t) = \frac{M_{pl}^2}{8\pi H^2}\int^t dt\,\Delta V\left[\frac{1}{2r^2}\frac{d}{dr}\left(2v^3\frac{dv}{dr}\right)\right]$$

$$= \frac{M_{pl}^2}{8\pi H^2}\int^t dt\, v\frac{d}{dr}\left(v^3\frac{dv}{dr}\right)\Delta\theta\,\Delta\varphi\,\Delta t = \frac{M_{pl}^2}{8\pi H^2} v^3\frac{dv}{dr}\,\Delta\theta\,\Delta\varphi\,\Delta t$$

Dividing out the volume, we recover the expression described in the text, with its normalization successfully matched to the deSitter-space limit.

$$n(r) = \frac{M_{pl}^2}{16\pi^2 H^2}\left(\frac{v^2}{r^2}\frac{dv}{dr}\right) \qquad v^2 = \frac{2GM}{r} = \frac{2M}{M_{pl}^2\, r}$$

Once the source radius R is defined, we can solve for n(r) as function of radius:

$$n(r) = \frac{M_{pl}^2}{32\pi^2 H^2}\left(\frac{v}{r}\right)^3 = \frac{M_{pl}^2}{32\pi^2 H^2}\left(\frac{2M}{M_{pl}^2 r^3}\right)^{3/2}$$

$$= \frac{\sqrt{2}}{16\pi^2 H^2 M_{pl}}\left(\frac{R}{r}\right)^{9/2}\left(\frac{M}{R^3}\right)^{3/2}$$

The critical density, along with the critical distance r_c from the center of the source is then

$$n_c \equiv M_{pl}^3 = \frac{\sqrt{2}}{16\pi^2 H^2 M_{pl}}\left(\frac{R}{r_c}\right)^{9/2}\left(\frac{M}{R^3}\right)^{3/2}$$

The ~~proton~~ Source mass and the scale factor for the radius of the source are defined as

$$M = m_p A \cong 5A\,\Lambda_{QCD}$$

$$R = R_0 A^{1/3}$$

$$R_0 = 1.2 \times 10^{-13}\ cm. = 1.2\,\Lambda_{QCD}^{-1}$$

This gives

$$\frac{M}{R^3} = \frac{5A\,\Lambda_{QCD}}{(1.2)^3\,\Lambda_{QCD}^{-3}\,A} \cong 3\,\Lambda_{QCD}^4$$

Consequently,

$$16\pi^2 H^2 M_{pl}^4 \cong 3\sqrt{6}\left(\frac{R}{r_c}\right)^{9/2}\Lambda_{QCD}^6$$

The critical radius is therefore

$$\left(\frac{r_c}{R}\right)^{9/2} = 3\sqrt{6}\left(\frac{\Lambda_{QCD}^3}{4\pi H M_{pl}^2}\right)^2$$

$$\frac{r_c}{R} = (7.4)^{2/9}\left(\frac{\Lambda_{QCD}^3}{4\pi H M_{pl}^2}\right)^{4/9} \approx 2.5$$

IV. Rift and Subduction Zones

The PG language used in the preceding section for the neutron star could also have been used for the cosmological example. And the two descriptions can in principle be synthesized within the PG language, just by using the PG formalism for the combined Schwarzschild-deSitter (Kottler) geometry. However there is a distinction to be made. In the case of the dark energy problem, the PG velocity field represented flow outward from the origin, while in the Schwarzschild problem it is natural (especially when dealing with black holes) to use an inward-going flow. This problem has bothered me for a long time. But in the context of classical Einstein general relativity, it has an easy solution (I thank Bill Unruh for teaching me this). One simply uses two separate coordinate patches to cover the full spacetime. In principle any spherical surface with arbitrary radius r will suffice in this regard as a boundary. Inside the sphere, inflow is chosen; outside the sphere the outflow of the expanding universe is chosen. But there is a natural choice (more on this later), which is where $v' = dv/dr$ vanishes.

With such a choice, it is easy to see that the curvature tensor is dominated outside the rift surface by dark energy and is dominated inside the rift surface by the matter source. The two contributions to the Riemann tensor R are additive, because R is linear in u and its derivatives, while u itself is additive; recall

$$u = v^2 = \frac{2GM}{r} + H^2 r^2$$

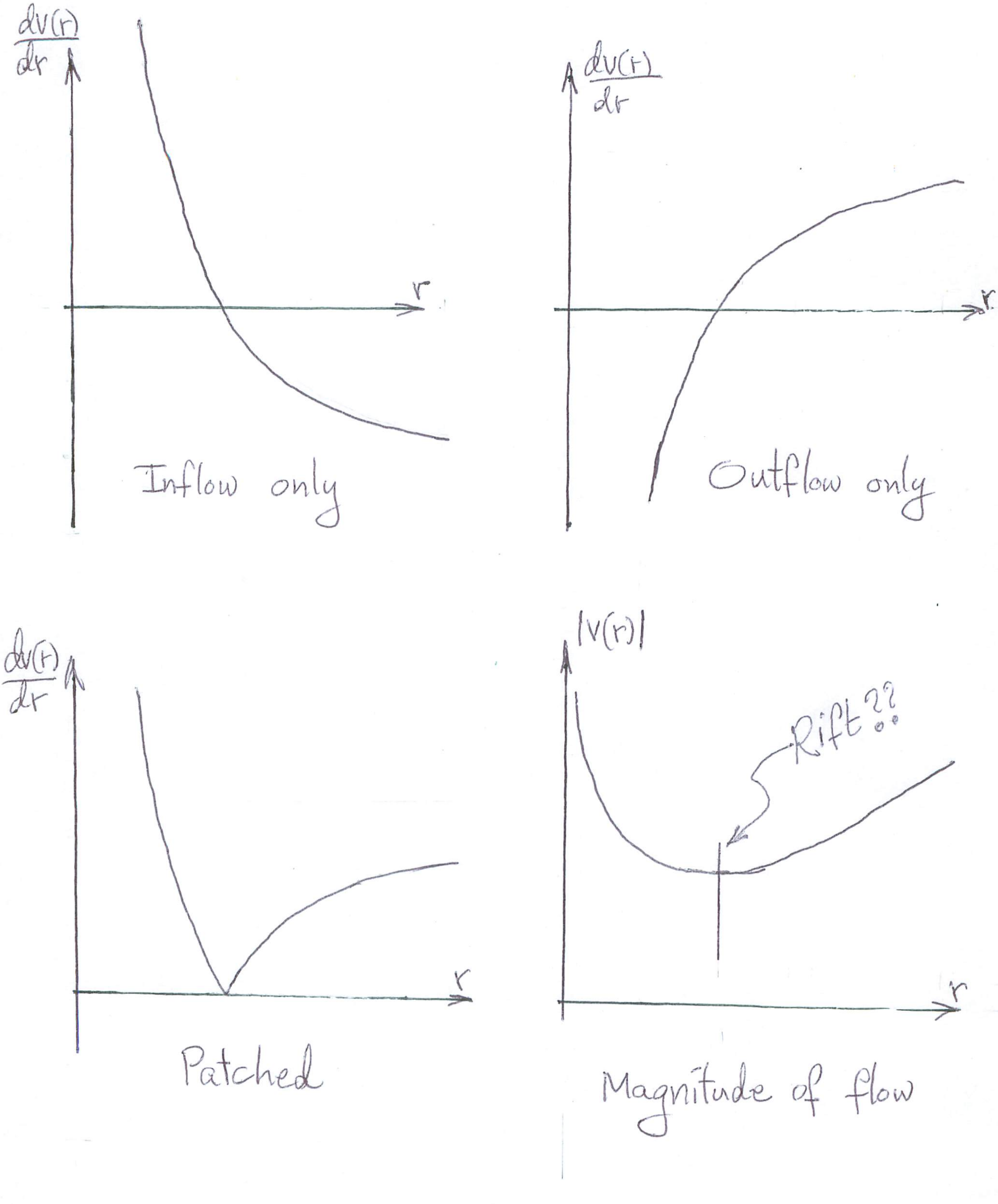
When making coordinate patches as described above, one must take care that matching conditions are satisfied. This boils down in this case to the statement that the Einstein tensor, and thereby the energy-momentum tensor , should not be modified on the boundary surface. This is assured, because all we have done is to reverse the sign of the velocity v in the interior region, and every component of the Riemann tensor is even in v .

Nevertheless, this patch seems to affect the interpretation we have made of the "topological density" n(r, t) , which is odd in v . The situation is depicted in the figures shown on the next page. By patching, we succeed in keeping n positive-semidefinite for all radii, while an unpatched coordinatization would exhibit a change of sign.

What this means, if anything, is totally unclear. Maybe this is evidence that this whole MM story line in these notes is some kind of nonsense. I obviously choose not to jump to that conclusion, but pursue the alternative. This surface from which the "topological charge" seems to emerge I will call a rift surface, since it seems slightly akin to what happens deep underwater in the mid-Atlantic. It has curious properties. For example, if one assigns an Unruh temperature to the acceleration of the PG flow

$$2\pi T_{Unruh} \equiv \frac{dv}{dt} = v\frac{dv}{dr} = \frac{u'}{2}$$

then it vanishes on the rift surface, but does not change sign. Also, if one uses the loop quantum gravity formalism (which is essentially the first-order formalism for gravity underlying the MM extension), the triad of gauge potentials conjugate to the electric-field triad in that formalism becomes degenerate at



The PG flow in the presence of coordinate patching.

the rift surface and changes from a right-handed triad to a left-handed triad as one crosses the surface unless one reverses the direction of flow. This occurs because the LQG triad describes the extrinsic curvature tensor K , which in turn is defined in the PG description by the symmetric gradient of the PG velocity field (see the Fischer-Visser paper, arXiv 0205139, for a good discussion of this). By construction, K becomes degenerate on the rift surface.

I have no good idea of whether any of this means very much. In the context of classical GR, I doubt that it does. But in the context of the quantum theory, or of the properties of the MM formalism that go beyond the Einstein equations of motion, or of the possible interpretation of gravity (and other standard model forces) as being emergent, there is in my opinion the possibility that this kind of thing has some significance.

It is especially in the context of emergence, as exemplified by the analog systems for gravity found in condensed matter phenomena, that I am tempted to think of the possibility that the rift suface (or a rift zone surrounding the rift surface) might contain beyond-the-standard-model observable physics. If gravitons are non-fundamental composite objects, and if sacred symmetries such as gauge invariance, general covariance, and Lorentz invariance are only approximate, then perhaps one might somehow see effects in that interesting region of spacetime. And it should be pointed out that the PG description underlying the above argument is ubiquitous in condensed-matter analog-gravity descriptions.

From the point of view of astrophysics, the rift surface which divides dark-energy-dominated spacetime from dark-matter-dominated spacetime evolves with the cosmological evolution. In the past, at redshifts of order one, small bubbles of deSitter spacetime started to emerge from the early state, which was fully matter-dominated. In the future, after a few efolds of exponential dark-energy-driven expansion have occurred, there will be isolated islands of gravitationally bound matter within a vast sea of deSitter spacetime. Therefore the rift-surface geometry will again consist of an ensemble of isolated bubbles surrounding the matter islands. But at present we are in the midst of a "phase transition" characterized by long-range connectivity of the rift surface. It is my impression that the piece of this rift surface that is nearest to us is beyond the local group, some tens of megaparsecs away. But this is a rather uneducated surmise on my part.

If "spacetime topology" is being produced at the rift surfaces, and flows away with the PG flow, where does it end up? In the outward directions it simply ends up contributing to the deSitter vacuum structure along with the dark energy density. The inward flow, say toward a neutron star, continues until the critical Planckian density is reached, at which point we expect something new has to happen. In the interpretation we have made, six extra dimensions of QCD scale "open up" in this region (i.e. must be taken into account in the MM action), and the topology flows into those dimensions. Therefore such spacetime regions, where the "topological density" $n(r,t)$ becomes Planckian, might well be dubbed "subduction zones". Just as for the rift zone, this might (if we are very lucky) also be a spacetime region where subtle, unexpected, beyond-the-standard-model phenomena occur.

There is a very good chance that the contents of this section are total nonsense. These words are meant to be provocative, not explanatory.