Math 2471 - Calc 3

Vector Functions

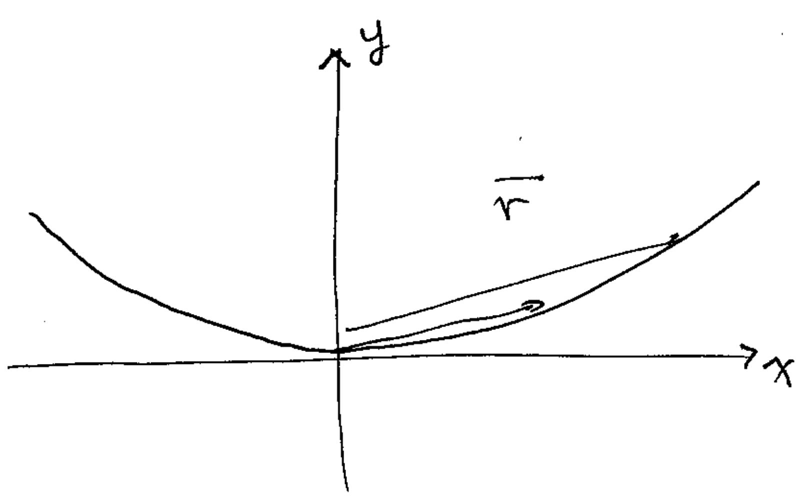
$\bar{r}(t) = \langle f(t), g(t) \rangle$

or $\bar{r}(t) = \langle f(t), g(t), h(t) \rangle$

& $x = f(t)$, $y = g(t)$ (and/or $z = h(t)$)

defines a space curve.

ex $\bar{r} = \langle t, \frac{1}{2}t^2 \rangle$



$\leftarrow$ $x = t$, $y = \frac{1}{2}t^2$

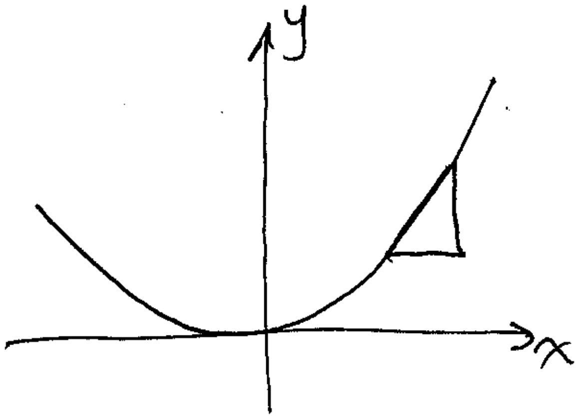
$\Downarrow$

$y = \frac{1}{2}x^2$

the derivative $\bar{r}(t)$ is tangent to

the space curve

& $\bar{r}'(t) = \langle 1, t \rangle$



if $y = \frac{1}{2}x^2$

$\frac{dy}{dx} = x$

slope is for some x value

which we see this in $\bar{r}'(t)$.

We define the unit normal

$$\bar{T} = \frac{\bar{r}'(t)}{\|\bar{r}'(t)\|}$$

so here $\|\bar{r}'\| = \sqrt{1+t^2}$

$$\bar{T} = \left\langle \frac{1}{\sqrt{1+t^2}}, \frac{t}{\sqrt{1+t^2}} \right\rangle$$

We defined another vector - the unit normal vector $\overline{N}$ where

$$\overline{N} = \frac{\overline{T}'}{\|\overline{T}'\|}$$

so continuing with this example

$$\overline{T}' = \left\langle -\frac{t}{(1+t^2)^{3/2}}, \frac{1}{(1+t^2)^{3/2}} \right\rangle$$

$$\|\overline{T}'\| = \sqrt{\frac{t^2}{(1+t^2)^3} + \frac{1}{(1+t^2)^3}}$$

$$= \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2}$$

so

$$\overline{N} = \left\langle \frac{-t}{\sqrt{1+t^2}}, \frac{1}{\sqrt{1+t^2}} \right\rangle$$

& $\overline{T} \cdot \overline{N} = 0$ ✓

Aside

$$x = (1+t^2)^{-1/2}$$

$$x' = -\frac{1}{2}(1+t^2)^{-3/2} \cdot 2t$$

$$= \frac{-t}{(1+t^2)^{3/2}}$$

$$y = \frac{t}{\sqrt{1+t^2}}$$

$$y' = \frac{1\sqrt{1+t^2} - t\,\frac{1}{\cancel{2}}(1+t^2)^{-1/2}\cancel{2}t}{1+t^2}$$

$$= \frac{\sqrt{1+t^2} - \frac{t^2}{\sqrt{1+t^2}}}{1+t^2}$$

$$= \frac{1}{(1+t^2)^{3/2}}$$

We also want to define one more vector perpendicular to both $\bar{T}$ & $\bar{N}$ - Called the binormal

$$\bar{B} = \bar{T} \times \bar{N}$$ (this will be in 3-D)

there is a property of cross products that

$$\|\bar{T} \times \bar{N}\| = \|\bar{T}\| \, \|\bar{N}\| \sin\theta$$

& since $\bar{T} \perp \bar{N}$ then $\theta = \pi/2$ & since $\|\bar{T}\| = 1$ and $\|\bar{N}\| = 1$ so

$$\|\bar{B}\| = \|\bar{T}\| \, \|\bar{N}\| = 1$$

so $\bar{B}$ will automatically be a unit normal

ex $\quad \bar{r} = \langle \cos t, \sin t, t \rangle$

$$\bar{r}' = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\bar{r}'\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\bar{T} = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, 1 \right\rangle$$

$$\bar{T}' = \left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right\rangle$$

$$\|T'\| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \frac{1}{\sqrt{2}}$$

$$\bar{N} = \langle -\cos t, -\sin t, 0 \rangle$$

$$\bar{B} = \bar{T} \times \bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\cos t}{\sqrt{2}} & 1 \\ -\sin t & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -\frac{\sin t}{\sqrt{2}} & 1 \\ -\cos t & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} \\ -\cos t & -\sin t \end{vmatrix} \bar{k}$$

$$= \left\langle \sin t, -\cos t, \frac{1}{\sqrt{2}} \right\rangle$$

so $\bar{T} = \langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, 1 \rangle$

$\bar{N} = \langle -\cos t, -\sin t, 0 \rangle$

$\bar{B} = \langle \sin t, -\cos t, \frac{1}{\sqrt{2}} \rangle$

$\bar{T} \cdot \bar{N} = \frac{\sin t \cos t}{\sqrt{2}} - \sin t \frac{\cos t}{\sqrt{2}} = 0$ ✓

$\bar{T} \cdot \bar{B} = -\frac{\sin^2 t}{\sqrt{2}} - \frac{\cos^2 t}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$ ✓

$\bar{N} \cdot \bar{B} = -\cos t \sin t + \sin t \cos t = 0$ ✓

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