Math 4315 – PDE's

Method of Characteristics – Continued

Given $a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$

M of C $\quad \dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{du}{c}$

Ex 1 $\quad x\,u_x + (x+2y)\,u_y = u \qquad u(x,x) = x^2$

M of C $\quad \dfrac{dx}{x} = \dfrac{dy}{x+2y} = \dfrac{du}{u}$

1st pair $\quad \dfrac{dx}{x} = \dfrac{dy}{x+2y}$ or $\dfrac{dy}{dx} = \dfrac{x+2y}{x}$ linear

$$\frac{dy}{dx} - \frac{2y}{x} = 1 \qquad \mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x^2} \Rightarrow \frac{y}{x^2} = -\frac{1}{x} + c_1$$

2nd pair $\quad \dfrac{dx}{x} = \dfrac{du}{u} \Rightarrow \ln|x| = \ln|u| - \ln C_2$

$\Rightarrow \quad C_2 = u/x$

sol^n $\quad C_2 = f(C_1) \Rightarrow \quad \dfrac{u}{x} = f\left(\dfrac{y}{x^2} + \dfrac{1}{x}\right)$

Now BC $\quad u(x,x) = x^2$

$\Rightarrow \quad \dfrac{x^2}{x} = f\left(\dfrac{x}{x^2} + \dfrac{1}{x}\right) = f\left(\dfrac{2}{x}\right)$

So $\quad f\left(\dfrac{2}{x}\right) = x$

Let $\quad \dfrac{2}{x} = \lambda \Rightarrow \quad x = \dfrac{2}{\lambda}$

$\Rightarrow \quad f(\lambda) = \dfrac{2}{\lambda} \quad \leftarrow$ now we know f

$\dfrac{u}{x} = f\left(\dfrac{y+x}{x^2}\right) = \dfrac{2}{\dfrac{y+x}{x^2}}$

$$\Rightarrow \boxed{u = \frac{2x^3}{x+y} \quad \text{Sol}^n}$$

Ex 2 $\quad y u_x + (u-x) u_y = y \qquad u(x,0) = 2x$

M of C $\quad \dfrac{dx}{y} = \dfrac{dy}{u-x} = \dfrac{du}{y}$

1st pair $\quad \dfrac{dx}{y} = \dfrac{du}{y} \Rightarrow x = u - c_1 \Rightarrow c_1 = u - x$

2nd pair $\quad \dfrac{dy}{u-x} = \dfrac{du}{y} \Rightarrow y\,dy = (u-x)\,du$

$$= c_1\,du$$

$$\Rightarrow \quad \frac{y^2}{2} = c_1 u + c_2$$

$$\Rightarrow \quad c_2 = \frac{y^2}{2} - (u-x)x$$

Solⁿ (i) $\quad \dfrac{y^2}{2} - (u-x)x = f(u-x)$ $\quad \leftarrow$ maybe easier

(ii) $\quad u - x = g\left(y^2/2 - (u-x)x \right)$

Sub B.C. $y=0,\ u=2x$

$$\frac{y^2}{2}-(u-x)x=f(u-x)$$

$$\Rightarrow\quad \frac{0}{2}-(2x-x)x=f(2x-x)$$

$$\Rightarrow\quad -x^2=f(x)$$ ← we know f now

$$\boxed{\text{Sol}^n\quad \frac{y^2}{2}-(u-x)x=-(u-x)^2}$$

we could solve for u be we will leave it implict

Ex3 $(y+u)u_x+(x-u)u_y=x+y$

M.C $$\frac{dx}{y+u}=\frac{dy}{x-u}=\frac{du}{x+y}$$

they are all coupled

we will choose x, y or u as our indep variable

so $\frac{dx}{y+u} = \frac{du}{x+y} \Rightarrow \frac{dx}{du} = \frac{y+u}{x+y}$

$\frac{dy}{x-u} = \frac{du}{x+y} \Rightarrow \frac{dy}{du} = \frac{x-u}{x+y}$

Adding these two eqn's

$$\frac{dx}{du} + \frac{dy}{du} = \frac{y+\cancel{u}+x-\cancel{u}}{x+y} = \frac{x+y}{x+y} = 1$$

$$\Rightarrow \quad x+y = u - c_1 \quad \text{or} \quad \boxed{c_1 = u - x - y}$$

Now we can use this to elliminate x, y or u

Here we choose x so

$$\frac{dy}{\cancel{u}-c_1-y-\cancel{u}} = \frac{du}{u-c_1} \Rightarrow -\frac{dy}{y+c_1} = \frac{du}{u-c_1}$$

$$-\ln|y+c_1| = \ln|u-c_1| - \ln C_2$$

$$\Rightarrow C_2 = (u-c_1)(y+c_1) \quad \Leftarrow \text{bring in } c_1$$

$$= (u-(u-x-y))(y+u-x-y)$$

$$= (x+y)(u-x)$$

so $\quad c_1 = u-x-y$

$\quad C_2 = (x+y)(u-x)$

sol^n $\quad (x+y)(u-x) = f(u-x-y)$

Yes - a harder example!