Math 4315 - PDE's Wed Oct-26/16

Last class

we saw

$$\frac{8}{\pi}\sin x + \frac{8}{27\pi}\sin 3x + \frac{8}{125\pi}\sin 5x$$

gave a good approx. to $f(x) = \pi x - x^2$

So we consider approx. function using sin & cos's

Like Taylor series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

where we look at derivatives, we do something similar

Fourier Series on $[-\pi, \pi]$

$$a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots \qquad = f(x) \qquad (1)$$
$$+ b_1\sin x + b_2\sin 2x + b_3\sin 3x + \dots$$

or

$$= a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

we need the following $\int$ formulas

1st $\displaystyle\int_{-\pi}^{\pi} \sin nx\,dx = 0$ $\displaystyle\int_{-\pi}^{\pi} \cos nx\,dx$

so integrate (1) $\displaystyle\int_{-\pi}^{\pi} — \,dx$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} (a_1 \cos x + a_2 \cos 2x + \cdots)\,dx$$

$$+ \int_{-\pi}^{\pi} (b_1 \sin x + b_2 \sin 2x + \cdots)\,dx$$

$$= a_0 x \Big|_{-\pi}^{\pi} = a_0 (\pi - (-\pi)) = 2\pi a_0$$

so $\displaystyle a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$

Next. multiply every thing by $\cos x$

$$f(x) \cos x = a_0 \cos x + a_1 \cos^2 x + a_2 \cos x \cos 2x + \cdots$$

$$+ b_1 \sin x \cos x + b_2 \sin 2x \cos x + \cdots$$

Now $\displaystyle\int_{-\pi}^{\pi} — \,dx$

$$\int_{-\pi}^{\pi} f(x)\cos x\,dx = a_0 \int_{-\pi}^{\pi} \cos x\,dx + a_1 \int_{-\pi}^{\pi} \cos^2 x\,dx$$

$$+ a_2 \int_{-\pi}^{\pi} \cos x \cos 2x\,dx + \cdots$$

$$+ b_1 \int_{-\pi}^{\pi} \sin x \cos x\,dx + b_2 \int_{-\pi}^{\pi} \sin 2x \cos x\,dx + \cdots$$

$$\int_{-\pi}^{\pi} f(x)\cos x\,dx = a_0 \cdot 0 + \pi a_1 + a_2 \cdot 0 \cdots$$

$$+ b_1 \cdot 0 + b_2 \cdot 0 \cdots$$

$$\Rightarrow \quad a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos x\,dx$$

similarly $a_2 = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\cos 2x\,dx$

$$a_3 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos 3x\,dx$$

$$\vdots$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

Now multiply by $\sin x$

$$f(x)\sin x = a_0 \sin x + a_1 \cos x \sin x + a_2 \cos 2x \sin x + \cdots$$
$$+ b_1 \sin^2 x + b_2 \sin x \sin 2x + \cdots$$

Now $\displaystyle\int_{-\pi}^{\pi} \text{—}\, dx$

$$\int_{-\pi}^{\pi} f(x)\sin x\,dx = a_0 \int_{-\pi}^{\pi} \sin x\,dx + a_1 \int_{-\pi}^{\pi} \cos x \sin x\,dx + \cdots$$
$$+ b_1 \int_{-\pi}^{\pi} \sin^2 x\,dx + \cdots$$

$$= a_0 \cdot 0 + a_1 \cdot 0 + \cdots$$
$$+ b_1 \pi + b_2 \cdot 0 \cdots$$

$$\Rightarrow \quad b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin x\,dx$$

$\vdots$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Now $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

So put ½ directly into Fourier's series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

Fourier Series

Consider the series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

with the use of the following integral formulas

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0, \qquad \int_{-\pi}^{\pi} \sin nx dx = 0,$$

$$\int_{-\pi}^{\pi} \cos nx \ \cos mx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \ \sin mx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \ \cos mx dx = 0.$$

we obtain

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$