

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 1

#### Question:

Differentiate with respect to  $x$ .

$\sinh 2x$

#### Solution:

$$\frac{d}{dx}(\sinh 2x) = 2 \cosh 2x$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 2

#### Question:

Differentiate with respect to  $x$ .  
 $\cosh 5x$

#### Solution:

$$\begin{aligned}\frac{d}{dx}(\cosh 5x) &= \frac{-1}{(\cosh 2x)^2} \times 2 \sinh 2x \\ &= -2 \frac{\sinh 2x}{\cos 2x} \times \frac{1}{\cos 2x} \\ &= -2 \tan 2x \operatorname{sech} 2x\end{aligned}$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 3

#### Question:

Differentiate with respect to  $x$ .  
 $\tanh 2x$

#### Solution:

$$\frac{d}{dx}(\tanh 2x) = 2\operatorname{sech}^2 2x$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 4

#### Question:

Differentiate with respect to  $x$ .  
 $\sinh 3x$

#### Solution:

$$\frac{d}{dx}(\sinh 3x) = 3 \cosh 3x$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

Exercise A, Question 5

**Question:**

Differentiate with respect to  $x$ .  
 $\coth 4x$

**Solution:**

$$\frac{d}{dx}(\coth 4x) = -4\operatorname{cosech}^2 4x$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

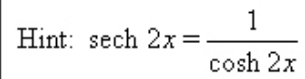
### Differentiation

#### Exercise A, Question 6

#### Question:

Differentiate with respect to  $x$ .

$\operatorname{sech} 2x$


$$\text{Hint: } \operatorname{sech} 2x = \frac{1}{\cosh 2x}$$

#### Solution:

$$\begin{aligned} \frac{d}{dx}(\operatorname{sech} 2x) &= \frac{-1}{(\cosh 2x)^2} \times 2 \sinh 2x \\ &= -2 \frac{\sinh 2x}{\cosh 2x} \times \frac{1}{\cosh 2x} \\ &= -2 \tanh 2x \operatorname{sech} 2x \end{aligned}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 7

#### Question:

Differentiate with respect to  $x$ .

$$e^{-x} \sinh x$$

#### Solution:

$$\begin{aligned} \frac{d}{dx} (e^{-x} \sinh x) &= -e^{-x} \sinh x + e^{-x} \cosh x \\ &= e^{-x} (\cosh x - \sinh x) \end{aligned}$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 8

#### Question:

Differentiate with respect to  $x$ .  
 $x \cosh 3x$

#### Solution:

$$\frac{d}{dx}(x \cosh 3x) = \cosh 3x + 3x \sinh 3x$$



# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 9

#### Question:

Differentiate with respect to  $x$ .

$$\frac{\sinh x}{3x}$$

#### Solution:

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sinh x}{3x}\right) &= \frac{\cosh x}{3x} - \frac{\sinh x}{3x^2} \\ &= \frac{x \cosh x - \sinh x}{3x^2}\end{aligned}$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 10

#### Question:

Differentiate with respect to  $x$ .

$$x^2 \cosh 3x$$

#### Solution:

$$\begin{aligned}\frac{d}{dx}(x^2 \cosh 3x) &= 2x \cosh 3x + x^2 \times 3 \sinh 3x \\ &= x(2 \cosh 3x + 3x \sinh 3x)\end{aligned}$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 11

#### Question:

Differentiate with respect to  $x$ .  
 $\sinh 2x \cosh 3x$

#### Solution:

$$\begin{aligned}\frac{d}{dx}(\sinh 2x \cosh 3x) &= 2 \cosh 2x \cosh 3x + \sinh 2x \times 3 \sinh 3x \\ &= 2 \cosh 2x \cosh 3x + 3 \sinh 2x \sinh 3x\end{aligned}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 12

#### Question:

Differentiate with respect to  $x$ .  
 $\ln(\cosh x)$

#### Solution:

$$\begin{aligned}\frac{d}{dx}(\ln \cosh x) &= \frac{1}{\cosh x} \times \sinh x \\ &= \tanh x\end{aligned}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 13

#### Question:

Differentiate with respect to  $x$ .

$$\sinh x^3$$

#### Solution:

$$\frac{d}{dx}(\sinh x^3) = 3x^2 \cosh x^3$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 14

#### Question:

Differentiate with respect to  $x$ .

$$\cosh^2 2x$$

#### Solution:

$$\begin{aligned}\frac{d}{dx}(\cosh^2 2x) &= 2 \cosh 2x \cdot 2 \sinh 2x \\ &= 4 \cosh 2x \sinh 2x\end{aligned}$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 15

#### Question:

Differentiate with respect to  $x$ .  
 $e^{\cosh x}$

#### Solution:

$$\frac{d}{dx}(e^{\cosh x}) = \sinh x e^{\cosh x}$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

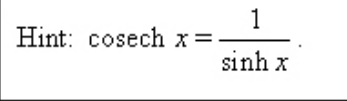
### Differentiation

#### Exercise A, Question 16

#### Question:

Differentiate with respect to  $x$ .

$\operatorname{cosech} x$



Hint:  $\operatorname{cosech} x = \frac{1}{\sinh x}$ .

#### Solution:

$$\begin{aligned}\frac{d}{dx}(\operatorname{cosech} x) &= \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{0 - 1 \times \cosh x}{\sinh^2 x} \\ &= -\operatorname{coth} x \operatorname{cosech} x\end{aligned}$$



# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 17

#### Question:

If  $y = a \cosh nx + b \sinh nx$ , where  $a$  and  $b$  are constants, prove that  $\frac{d^2y}{dx^2} = n^2y$ .

#### Solution:

$$y = a \cosh nx + b \sinh nx$$

Differentiate with respect to  $x$

$$\frac{dy}{dx} = an \sinh nx + nb \cosh nx$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= an^2 \cosh nx + bn^2 \sinh nx \\ &= n^2 (a \cosh nx + b \sinh nx) \end{aligned}$$

$$\frac{d^2y}{dx^2} = n^2y$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 18

#### Question:

Find the stationary values of the curve with equation  $y = 12\cosh x - \sinh x$ .

#### Solution:

$$y = 12\cosh x - \sinh x$$

$$\frac{dy}{dx} = 12\sinh x - \cosh x$$

At stationary values  $\frac{dy}{dx} = 0$

$$0 = 12\sinh x - \cosh x$$

$$\cosh x = 12\sinh x$$

$$\frac{1}{12} = \tanh x$$

$$x = \tanh^{-1} \frac{1}{12}$$

$$x = 0.0835$$

The stationary value is therefore  $y = 12\cosh 0.0835 - \sinh 0.0835$   
 $= 12.13$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 19

#### Question:

Given that  $y = \cosh 3x \sinh x$ , find  $\frac{d^2y}{dx^2}$ .

#### Solution:

$$y = \cosh 3x \sinh x$$

$$\frac{dy}{dx} = 3 \sinh 3x \sinh x + \cosh 3x \cosh x$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= 9 \cosh 3x \sinh x + 3 \sinh 3x \cosh x + 3 \sinh 3x \cosh x + \cosh 3x \sinh x \\ &= 10 \cosh 3x \sinh x + 6 \sinh 3x \cosh x \\ &= 2(5 \cosh 3x \sinh x + 3 \sinh 3x \cosh x)\end{aligned}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise A, Question 20

#### Question:

Find the equation of the tangent and normal to the hyperbola  $\frac{x^2}{256} - \frac{y^2}{16} = 1$  at the point  $(16 \cosh q, 4 \sinh q)$ .

#### Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dq}}{\frac{dx}{dq}} = \frac{4 \cosh q}{16 \sinh q} = \frac{\cosh q}{4 \sinh q}$$

Equation of tangent

$$y - 4 \sinh q = \frac{\cosh q}{4 \sinh q} (x - 16 \cosh q)$$

$$4y \sinh q - 16 \sinh^2 q = x \cosh q - 16 \cosh^2 q$$

$$4y \sinh q - x \cosh q = 16(\sinh^2 q - \cosh^2 q)$$

$$4y \sinh q - x \cosh q = -16$$

$$\text{or } x \cosh q - 4y \sinh q = 16$$

Equation of normal

$$y - 4 \sinh q = \frac{-4 \sinh q}{\cosh q} (x - 16 \cosh q)$$

$$\text{i.e. } y \cosh q - 4 \sinh q \cosh q = -4x \sinh q + 64 \sinh q \cosh q$$

$$\text{i.e. } y \cosh q + 4x \sinh q = 68 \sinh q \cosh q$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 1

#### Question:

Differentiate

a  $\operatorname{arcosh} 2x$

b  $\operatorname{arsinh}(x+1)$

c  $\operatorname{artanh} 3x$

d  $\operatorname{arsech} x$

e  $\operatorname{arcosh} x^2$

f  $\operatorname{arcosh} 3x$

g  $x^2 \operatorname{arcosh} x$

h  $\operatorname{arsinh} \frac{x}{2}$

i  $e^{x^3} \operatorname{arsinh} x$

j  $\operatorname{arsinh} x \operatorname{arcosh} x$

k  $\operatorname{arcosh} x \operatorname{sech} x$

l  $x \operatorname{arcosh} 3x$

#### Solution:

a Let  $y = \operatorname{arcosh} 2x$  then  $\cosh y = 2x$

Differentiate with respect to  $x$

$$\sinh y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{\sinh y}$$

$$= \frac{2}{\sqrt{\cosh^2 y - 1}} \text{ but } \cosh y = 2x$$

$$\text{so } \frac{dy}{dx} = \frac{2}{\sqrt{4x^2 - 1}}$$

b Let  $y = \operatorname{arsinh}(x+1)$  then  $\sinh y = x+1$

$$\cosh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$= \frac{1}{\sqrt{\sinh^2 y + 1}} \text{ but } \sinh y = x+1$$

$$\text{so } \frac{dy}{dx} = \frac{1}{\sqrt{(x+1)^2 + 1}}$$

c Let  $y = \operatorname{artanh} 3x$

$$\tanh y = 3x$$

$$\operatorname{sech}^2 y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\operatorname{sech}^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1 - \tanh^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1 - 9x^2}$$

**d** Let  $y = \operatorname{arsech} x$

$$\operatorname{sech} y = x$$

$$\frac{1}{\cosh y} = x$$

$$1 = x \cosh y$$

Differentiate with respect to  $x$

$$0 = \cosh y + x \sinh y \frac{dy}{dx}$$

$$x \sinh y \frac{dy}{dx} = -\cosh y$$

$$\frac{dy}{dx} = \frac{-\cosh y}{x \sinh y}$$

$$= \frac{1}{x \tanh y}$$

$$= \frac{1}{x(1 - \operatorname{sech}^2 y)^{\frac{1}{2}}}$$

$$= \frac{-1}{x(1 - x^2)^{\frac{1}{2}}}$$

**e** Let  $y = \operatorname{arcosh} x^2$

$$\text{Let } t = x^2 \quad y = \operatorname{arcosh} t$$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 - 1}}$$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{x^4 - 1}}$$

**f**  $y = \operatorname{arcosh} 3x$

$$\text{Let } t = 3x \quad y = \operatorname{arcosh} t$$

$$\frac{dt}{dx} = 3 \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 - 1}}$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{9x^2 - 1}}$$

**g**  $y = x^2 \operatorname{arcosh} x$

$$\frac{dy}{dx} = 2x \operatorname{arcosh} x + \frac{x^2}{\sqrt{x^2 - 1}}$$

$$\mathbf{h} \quad y = \operatorname{arsinh} \frac{x}{2}$$

$$\text{Let } t = \frac{x}{2} \quad y = \operatorname{arsinh} t$$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2\sqrt{\left(\frac{x}{2}\right)^2+1}} \\ &= \frac{1}{\sqrt{x^2+4}} \end{aligned}$$

$$\mathbf{i} \quad y = e^{x^3} \operatorname{arsinh} x$$

$$\frac{dy}{dx} = 3x^2 e^{x^3} \operatorname{arsinh} x + \frac{e^{x^3}}{\sqrt{x^2+1}}$$

$$\mathbf{j} \quad y = \operatorname{arsinh} x \operatorname{arcosh} x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \operatorname{arcosh} x + \frac{1}{\sqrt{x^2-1}} \operatorname{arsinh} x$$

$$\mathbf{k} \quad y = \operatorname{arcosh} x \operatorname{sech} x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{x^2-1}} \operatorname{sech} x - \operatorname{arcosh} x \tanh x \operatorname{sech} x \\ &= \operatorname{sech} x \left( \frac{1}{\sqrt{x^2-1}} - \operatorname{arcosh} x \tanh x \right) \end{aligned}$$

$$\mathbf{l} \quad y = x \operatorname{arcosh} 3x$$

$$\frac{dy}{dx} = \operatorname{arcosh} 3x + x \times \frac{3}{\sqrt{9x^2-1}}$$



# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 2

#### Question:

Prove that

$$\mathbf{a} \quad \frac{d}{dx}(\operatorname{arcosh}x) = \frac{1}{\sqrt{x^2-1}}$$

$$\mathbf{b} \quad \frac{d}{dx}(\operatorname{artanh}x) = \frac{1}{1-x^2}$$

#### Solution:

$$\mathbf{a} \quad y = \operatorname{arcosh}x$$

$$\cosh y = x$$

$$\sinh y \frac{dy}{dx} = 1 \Rightarrow$$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}}$$

but  $\cosh y = x$  so

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$$

$$\mathbf{b} \quad y = \operatorname{artanh}x$$

$$\tanh y = x$$

$$\operatorname{sech}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-\tanh^2 y}$$

but  $\tanh y = x$  so

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 3

#### Question:

Given that  $y = \operatorname{artanh}\left(\frac{e^x}{2}\right)$ , prove that  $(4 - e^{2x})\frac{dy}{dx} = 2e^x$ .

#### Solution:

$$y = \operatorname{artanh} \frac{e^x}{2}$$

$$\text{Let } t = \frac{e^x}{2} \quad y = \operatorname{artanh} t$$

$$\frac{dt}{dx} = \frac{e^x}{2} \quad \frac{dy}{dt} = \frac{1}{1-t^2}$$

$$\text{Then } \frac{dy}{dx} = \frac{1}{1-t^2} \times \frac{e^x}{2}$$

$$= \frac{1}{1-\left(\frac{e^x}{2}\right)^2} \times \frac{e^x}{2}$$

$$= \frac{\frac{e^x}{2}}{\frac{4 - e^{2x}}{4}}$$

$$\frac{dy}{dx} = \frac{2e^x}{4 - e^{2x}}$$

$$(4 - e^{2x})\frac{dy}{dx} = 2e^x$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 4

#### Question:

Given that  $y = \operatorname{arsinh} x$ , show that

$$(1+x^2) \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

#### Solution:

$$y = \operatorname{arsinh} x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-\frac{1}{2}}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{1}{2}(x^2+1)^{-\frac{3}{2}} \times 2x \\ &= \frac{-x}{(x^2+1)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= \frac{-1(x^2+1)^{\frac{3}{2}} - \frac{3}{2}(x^2+1)^{\frac{1}{2}} \times 2x \times -x}{(x^2+1)^3} \\ &= \frac{3x^2(x^2+1)^{\frac{1}{2}} - (x^2+1)^{\frac{3}{2}}}{(x^2+1)^3} \\ &= \frac{3x^2}{(x^2+1)^{\frac{5}{2}}} - \frac{1}{(x^2+1)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} (x^2+1) \frac{d^3y}{dx^3} &= \frac{3x^2}{(x^2+1)^{\frac{3}{2}}} - \frac{1}{(x^2+1)^{\frac{1}{2}}} \\ &= -3x \frac{d^2y}{dx^2} - \frac{dy}{dx} \end{aligned}$$

$$\therefore (1+x^2) \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 5

Question:

If  $y = (\operatorname{arcosh} x)^2$ , find  $\frac{d^2y}{dx^2}$ .

Solution:

$$\begin{aligned}y &= (\operatorname{arcosh} x)^2 \\ \frac{dy}{dx} &= 2\operatorname{arcosh} x \times \frac{1}{\sqrt{x^2-1}} \\ &= 2(x^2-1)^{-\frac{1}{2}} \operatorname{arcosh} x \\ \frac{d^2y}{dx^2} &= -(x^2-1)^{-\frac{3}{2}} 2x \operatorname{arcosh} x + 2(x^2-1)^{-\frac{1}{2}} \times \frac{1}{\sqrt{x^2-1}} \\ &= \frac{-2x \operatorname{arcosh} x}{(x^2-1)^{\frac{3}{2}}} + \frac{2}{x^2-1}\end{aligned}$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise B, Question 6

##### Question:

Find the equation of the tangent at the point where  $x = \frac{12}{13}$  on the curve with equation  $y = \operatorname{artanh} x$ .

##### Solution:

$$y = \operatorname{artanh} x \quad x = \frac{12}{13} \quad y = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) = \frac{1}{2} \ln 25 = \ln 5$$

$$\frac{dy}{dx} = \frac{1}{1-x^2} = \frac{1}{1-\left(\frac{12}{13}\right)^2} = \frac{169}{25}$$

Tangent is

$$(y - \ln 5) = \frac{169}{25} \left( x - \frac{12}{13} \right)$$

$$25y - 25 \ln 5 = 169x - 156$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise C, Question 1

##### Question:

Given that  $y = \arccos x$  prove that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

##### Solution:

$$y = \arccos x$$

$$\cos y = x$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$$= \frac{-1}{\sqrt{1-\cos^2 y}}$$

since  $\cos y = x$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise C, Question 2

#### Question:

Differentiate with respect to  $x$

a  $\arccos 2x$

b  $\arctan \frac{x}{2}$

c  $\arcsin 3x$

d  $\operatorname{arccot} x$

e  $\operatorname{arcsec} x$

f  $\operatorname{arccosec} x$

g  $\arcsin \left( \frac{x}{x-1} \right)$

h  $\arccos x^2$

i  $e^x \arccos x$

j  $\arcsin x \cos x$

k  $x^2 \arccos x$

l  $e^{\arctan x}$

#### Solution:

**a** Let  $y = \arccos 2x$

Let  $t = 2x$   $y = \arccos t$

$$\text{then } \frac{dt}{dx} = 2 \quad \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{\sqrt{1-t^2}} \times 2 \\ &= \frac{-2}{\sqrt{1-4x^2}} \end{aligned}$$

**b** Let  $y = \arctan \frac{x}{2}$

Let  $t = \frac{x}{2}$   $y = \arctan t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{1+t^2}$$

$$\frac{dy}{dx} = \frac{1}{1+t^2} \times \frac{1}{2} = \frac{1}{2 \left( 1 + \frac{x^2}{4} \right)} = \frac{2}{4+x^2} \text{ or } \frac{2}{x^2+4}$$

**c** Let  $y = \arcsin 3x$

$\sin y = 3x$

$$\cos y \frac{dy}{dx} = 3$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3}{\cos y} = \frac{3}{\sqrt{1-\sin^2 y}} \\ &= \frac{3}{\sqrt{1-9x^2}} \\ &= \frac{3}{\sqrt{1-9x^2}} \end{aligned}$$



**d** Let  $y = \operatorname{arccot} x$

$$\cot y = x$$

$$-\operatorname{cosec}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y}$$

$$= \frac{-1}{1 + \cot^2 y}$$

$$= \frac{-1}{1 + x^2}$$

**e** Let  $y = \operatorname{arcsec} x$

$$\sec y = x$$

$$\sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$= \frac{1}{\sec y \sqrt{\sec^2 y - 1}}$$

$$= \frac{1}{x \sqrt{x^2 - 1}}$$

**f** Let  $y = \operatorname{arccosec} x$

$$\operatorname{cosec} y = x$$

$$-\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y \cot y}$$

$$= \frac{-1}{\operatorname{cosec} y \sqrt{(\operatorname{cosec}^2 y - 1)}}$$

$$= \frac{-1}{x \sqrt{x^2 - 1}}$$

**g** Let  $y = \arcsin\left(\frac{x}{x-1}\right)$

$$\sin y = \frac{x}{x-1}$$

$$\begin{aligned}
 \cos y \frac{dy}{dx} &= \frac{-1}{(x-1)^2} \\
 \frac{dy}{dx} &= \frac{1}{\cos y} \times \frac{-1}{(x-1)^2} \\
 &= \frac{1}{\sqrt{1-\frac{x^2}{(x-1)^2}}} \times \frac{-1}{(x-1)^2} \\
 &= \frac{1}{\sqrt{\frac{(x-1)^2 - x^2}{(x-1)^2}}} \times \frac{-1}{(x-1)^2} \\
 &= \frac{1}{\sqrt{1-2x}} \times \frac{-1}{(x-1)^2} \\
 &= \frac{-1}{(x-1)\sqrt{1-2x}}
 \end{aligned}$$

**h** Let  $y = \arccos x^2$

Let

$$t = x^2 \quad y = \arccos t$$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{-1}{\sqrt{1-t^2}} \times 2x \\
 &= \frac{-2x}{\sqrt{1-x^4}}
 \end{aligned}$$

**i** Let  $y = e^x \arccos x$

$$\begin{aligned}
 \frac{dy}{dx} &= e^x \arccos x - e^x \frac{1}{\sqrt{1-x^2}} \\
 &= e^x \left( \arccos x - \frac{1}{\sqrt{1-x^2}} \right)
 \end{aligned}$$

**j** Let  $y = \arcsin x \cos x$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} \cos x + \arcsin x \times -\sin x \\
 &= \frac{\cos x}{\sqrt{1-x^2}} - \sin x \arcsin x
 \end{aligned}$$

**k** Let  $y = x^2 \arccos x$

$$\begin{aligned}
 \frac{dy}{dx} &= 2x \arccos x - x^2 \times \frac{1}{\sqrt{1-x^2}} \\
 &= 2x \arccos x - \frac{x^2}{\sqrt{1-x^2}} \\
 &= x \left( 2 \arccos x - \frac{x}{\sqrt{1-x^2}} \right)
 \end{aligned}$$

**l** Let  $y = e^{\arctan x}$

$$\frac{dy}{dx} = \frac{e^{\arctan x}}{1+x^2}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise C, Question 3

#### Question:

If  $\tan y = x \arctan x$ , find  $\frac{dy}{dx}$ .

#### Solution:

$$\begin{aligned}\tan y &= x \arctan x \\ \sec^2 y \frac{dy}{dx} &= \arctan x + \frac{x}{1+x^2} \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \left( \arctan x + \frac{x}{1+x^2} \right) \\ &= \frac{1}{1+x^2 (\arctan x)^2} \left( \arctan x + \frac{x}{1+x^2} \right)\end{aligned}$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise C, Question 4

#### Question:

Given that  $y = \arcsin x$  prove that

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 0 \quad \text{[E]}$$

#### Solution:

$$y = \arcsin x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{0 - \frac{1}{2}(1-x^2)^{-\frac{1}{2}}x - 2x}{(\sqrt{1-x^2})^2}$$

$$= \frac{x(1-x^2)^{-\frac{1}{2}}}{(1-x^2)}$$

$$= \frac{x}{\sqrt{1-x^2}(1-x^2)}$$

$$(1-x^2) \frac{d^2y}{dx^2} = x \frac{dy}{dx}$$

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 0$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise C, Question 5

#### Question:

Find an equation of the tangent to the curve with equation  $y = \arcsin 2x$  at the point where  $x = \frac{1}{4}$ .

#### Solution:

$$y = \arcsin 2x \quad x = \frac{1}{4} \quad y = \arcsin\left(\frac{2}{4}\right) = \frac{\pi}{6}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} = \frac{2}{\sqrt{1-\frac{1}{4}}} = \frac{4}{\sqrt{3}}$$

Tangent is

$$\left(y - \frac{\pi}{6}\right) = \frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$$

$$\sqrt{3}y - \frac{\pi\sqrt{3}}{6} = 4x - 1$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 1

**Question:**

Given  $y = \cosh 2x$ , find  $\frac{dy}{dx}$ .

**Solution:**

$$y = \cosh 2x$$

$$\frac{dy}{dx} = 2 \sinh 2x$$

© Pearson Education Ltd 2009

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 2

#### Question:

Differentiate with respect to  $x$ .

a  $\operatorname{arsinh} 3x$

b  $\operatorname{arsinh} x^2$

c  $\operatorname{arcosh} \frac{x}{2}$

d  $x^2 \operatorname{arcosh} 2x$

#### Solution:

**a**  $y = \operatorname{arsinh} 3x$

Let  $t = 3x$   $y = \operatorname{arsinh} t$

$$\frac{dt}{dx} = 3 \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2+1}} \times 3$$

$$= \frac{3}{\sqrt{9x^2+1}}$$

**b**  $y = \operatorname{arsinh} x^2$

Let  $t = x^2$   $y = \operatorname{arsinh} t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2+1}} \times 2x$$

$$= \frac{2x}{\sqrt{x^4+1}}$$

**c**  $y = \operatorname{arcosh} \frac{x}{2}$

Let  $t = \frac{x}{2}$   $y = \operatorname{arcosh} t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2-1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2-1}} \times \frac{1}{2}$$

$$= \frac{1}{2\sqrt{\frac{x^2}{4}-1}} = \frac{1}{\sqrt{x^2-4}}$$

**d**  $y = x^2 \operatorname{arcosh} 2x$

$$\frac{dy}{dx} = 2x \operatorname{arcosh} 2x + x^2 \times \frac{2}{\sqrt{4x^2-1}}$$

$$= 2x \left( \operatorname{arcosh} 2x + \frac{x}{\sqrt{4x^2-1}} \right)$$



# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 3

##### Question:

Given that  $y = \arctan x$ , prove that

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

##### Solution:

$$y = \arctan x$$

$$\text{then } \tan y = x$$

$$\tan y = x$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\text{but } \sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\text{so } \frac{dy}{dx} = \frac{1}{1+x^2}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 4

#### Question:

Given that  $y = (\operatorname{arsinh} x)^2$  prove that

$$(1+x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 2 = 0$$

#### Solution:

$$y = (\operatorname{arsinh} x)^2$$

$$\frac{dy}{dx} = \frac{2(\operatorname{arsinh} x)^1}{\sqrt{x^2+1}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{2}{\sqrt{x^2+1}} \times \sqrt{x^2+1} - \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x \times 2\operatorname{arsinh} x}{(\sqrt{x^2+1})^2}$$

$$(x^2+1) \frac{d^2y}{dx^2} = 2 - 2x(x^2+1)^{-\frac{1}{2}} \operatorname{arsinh} x$$

$$= 2 - x \frac{dy}{dx}$$

$$(x^2+1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - 2 = 0$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 5

#### Question:

Given  $y = 5 \cosh x - 3 \sinh x$

a find  $\frac{dy}{dx}$

b find the minimum turning points.

#### Solution:

$$y = 5 \cosh x - 3 \sinh x$$

$$\frac{dy}{dx} = 5 \sinh x - 3 \cosh x$$

At maximum and minimum  $\frac{dy}{dx} = 0$

$$0 = 5 \sinh x - 3 \cosh x$$

$$3 \cosh x = 5 \sinh x$$

$$\frac{3}{5} = \tanh x$$

$$x = \operatorname{artanh} \frac{3}{5}$$

Use  $\operatorname{artanh} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)$

$$x = \frac{1}{2} \ln \left( \frac{8}{\frac{2}{5}} \right)$$

$$x = \frac{1}{2} \ln 4$$

$$= \ln 2$$

$$y = 6 \frac{1}{4} - 2 \frac{1}{4}$$

$$= 4$$

$\Rightarrow$  turning point is  $(\ln 2, 4)$

$$\frac{d^2y}{dx^2} = 5 \cosh x - 3 \sinh x = 4 \text{ at } x = \ln 2$$

$\therefore \frac{d^2y}{dx^2} > 0$  at  $(\ln 2, 4)$  so this point is a minimum

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 6

#### Question:

Given that  $y = (\arcsin x)^2$  show that

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - 2 = 0$$

#### Solution:

$$y = (\arcsin x)^2$$

$$\frac{dy}{dx} = 2(\arcsin x) \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{2 \times \frac{1}{\sqrt{1-x^2}} \times \sqrt{1-x^2} - 2 \arcsin x \times \frac{1}{2} (1-x^2)^{-\frac{3}{2}} \times -2x}{(1-x^2)}$$

$$(1-x^2) \frac{d^2y}{dx^2} = 2 + \frac{x \times 2 \arcsin x}{(1-x^2)^{\frac{1}{2}}}$$

$$= 2 + x \frac{dy}{dx}$$

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - 2 = 0$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 7

#### Question:

Differentiate  $\operatorname{arcosh}(\sinh 2x)$ .

#### Solution:

$$y = \operatorname{arcosh}(\sinh 2x)$$

$$\text{Let } t = \sinh 2x \quad y = \operatorname{arcosh} t$$

$$\frac{dt}{dx} = 2 \cosh 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 - 1}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{t^2 - 1}} \times 2 \cosh 2x \\ &= \frac{2 \cosh 2x}{\sqrt{\sinh^2 2x - 1}} \end{aligned}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 8

**Question:**

Given that  $y = x - \arctan x$ , prove that  $\frac{d^2y}{dx^2} = 2x \left(1 - \frac{dy}{dx}\right)^2$

**Solution:**

$$y = x - \arctan x$$

$$\frac{dy}{dx} = 1 - \frac{1}{1+x^2}$$

$$\frac{d^2y}{dx^2} = 0 - \frac{(0 - 2x)}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

$$= 2x \left(1 - \left(1 - \frac{1}{1+x^2}\right)\right)^2$$

$$\frac{d^2y}{dx^2} = 2x \left(1 - \frac{dy}{dx}\right)^2$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 9

Question:

Differentiate  $\arcsin \frac{x}{\sqrt{1+x^2}}$ .

Solution:

$$y = \arcsin \frac{x}{\sqrt{1+x^2}}$$

$$\text{Let } t = \frac{x}{\sqrt{1+x^2}}$$

$$y = \arcsin t$$

$$\frac{dt}{dx} = \frac{1(1+x^2)^{\frac{1}{2}} - \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \times 2x \times x}{(1+x^2)} \quad \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-t^2}} \left( \frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{1+x^2} \right)$$

$$= \frac{1}{\sqrt{1 - \frac{x^2}{1+x^2}}} \left( (1+x^2)^{-\frac{1}{2}} \frac{[1+x^2 - x^2]}{(1+x^2)} \right)$$

$$= \frac{1}{\sqrt{\frac{1}{1+x^2}}} \frac{[1]}{(1+x^2)^{\frac{1}{2}}} = \frac{1}{x^2+1}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 10

#### Question:

Show that the curve with equation  $y = \operatorname{sech} x$  has  $\frac{d^2y}{dx^2} = 0$  at the point where  $x = \pm \ln p$  and state a value of  $p$ .

#### Solution:



$$y = \operatorname{sech} x$$

$$\frac{dy}{dx} = -\tanh x \operatorname{sech} x$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \operatorname{sech}^2 x \operatorname{sech} x + \tanh x (-\tanh x \operatorname{sech} x) \\ &= \operatorname{sech}^3 x - \operatorname{sech} x \tanh^2 x \\ &= \operatorname{sech} x (\operatorname{sech}^2 x - \tanh^2 x) \\ &= \operatorname{sech} x (1 - \tanh^2 x - \tanh^2 x) \\ &= \operatorname{sech} x (1 - 2 \tanh^2 x) \end{aligned}$$

$$\text{When } \frac{d^2y}{dx^2} = 0$$

$$0 = \operatorname{sech} x (1 - 2 \tanh^2 x)$$

$$\text{so } \tanh^2 x = \frac{1}{2} \Rightarrow \tanh x = \pm \frac{1}{\sqrt{2}}$$

$$x = \operatorname{artanh} \pm \frac{1}{\sqrt{2}} = \pm \operatorname{artanh} \left( \frac{1}{\sqrt{2}} \right)$$

$$x = \pm \frac{1}{2} \ln \left( \frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} \right)$$

$$= \pm \frac{1}{2} \ln \left( \frac{\frac{\sqrt{2}+1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \right)$$

$$= \pm \frac{1}{2} \ln \left( \frac{\sqrt{2}+1}{\sqrt{2}-1} \right)$$

$$= \pm \frac{1}{2} \ln \left( \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} \right)$$

$$= \pm \frac{1}{2} \ln \left( \frac{(\sqrt{2}+1)^2}{2-1} \right)$$

$$= \pm \frac{1}{2} \ln (\sqrt{2}+1)^2$$

$$= \pm \ln (\sqrt{2}+1) \quad p = \sqrt{2}+1 \text{ (Note } p = \sqrt{2}-1 \text{ is also acceptable.)}$$

# Solutionbank FP3

## Edexcel AS and A Level Modular Mathematics

### Differentiation

#### Exercise D, Question 11

#### Question:

Find the equation of the tangent and normal to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  at the point  $(a \cosh q, b \sinh q)$ .

#### Solution:

$$x = a \cosh q \quad y = b \sinh q$$

$$\frac{dy}{dx} = \frac{b \cosh q}{a \sinh q}$$

$$\text{Equation of tangent } y - b \sinh q = \frac{b \cosh q}{a \sinh q} (x - a \cosh q)$$

$$ay \sinh q - ab \sinh^2 q = xb \cosh q - ab \cosh^2 q$$

$$ay \sinh q - xb \cosh q + ab(\cosh^2 q - \sinh^2 q) = 0$$

$$ay \sinh q - xb \cosh q + ab = 0$$

$$\text{Equation of normal } y - b \sinh q = -\frac{a \sinh q}{b \cosh q} (x - a \cosh q)$$

$$by \cosh q - b^2 \sinh q \cosh q = -ax \sinh q + a^2 \sinh q \cosh q$$

$$ax \sinh q + by \cosh q - \sinh q \cosh q (a^2 + b^2) = 0$$