Math 1497 - Calc 2

Trig Integrals

$$\int \tan^m x \sec^n x \, dx$$

Cases
(i) n even $n \geq 2$
(ii) m odd
(iii) $\tan x$ only
(iv) $\sec x$ only

Ex 1 $\int \tan x \sec^2 x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$\int u \, du = \frac{u^2}{2} + C$

$= \frac{1}{2} \tan^2 x + C$

Ex 2 $\int \tan^2 x \sec^4 x \, dx$

$$\int \tan^2 x \sec^2 x \sec^2 x \, dx$$

$u = \tan x$ Now If $\tan^2 x = \sec^2 x$

$du = \sec^2 x \, dx$ $\sec^2 x = 1 + u^2$

So $\int u^2 (1+u^2) \, du$

$$\int u^2 + u^4 \, du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

So when n is even pull off

2 sec's and let $u = \tan x$

Case (ii) m odd & there's at least 1 $\sec x$

Ex 3 $\int \tan^3 x \sec x \, dx$

$\int \tan^2 x \sec x \tan x \, dx$

let $u = \sec x$ $du = \sec x \tan x \, dx$

$1 + \tan^2 x = \sec^2 x \Rightarrow \tan^2 x = u^2 - 1$

$\int (u^2 - 1) du = \frac{u^3}{3} - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

Ex 4 $\int \tan^3 x \sec^3 x \, dx$

$\int \tan^2 x \sec^2 x \sec x \tan x \, dx$

$u = \sec x$ $du = \sec x \tan x \, dx$

$$\tan^2 x = \sec^2 x - 1 = u^2 - 1$$

$$\int (u^2-1)u^2\,du = \int u^4 - u^2\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$$

Case (ii) $\int \tan^m x\,dx$ (only $\tan x$)

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \qquad \begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}$$

$$= \int -\frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

$$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx$$

$$= \tan x - x + C$$

$$\int \tan^3 x\,dx = \int \tan x \tan^2 x\,dx$$

$$= \int \tan x \left(\sec^2 x - 1\right) dx$$

$$= \int \tan x \sec^2 x\,dx - \int \tan x\,dx$$

$u = \tan x$

$du = \sec^2 x\,dx$ — " "

$\int u\,du$ — " "

" "

$= \frac{u^2}{2}$ —

$$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$$

$$\int \tan^4 x\,dx = \int \tan^2 x \cdot \underbrace{\tan^2 x}_{\sec^2 x - 1}\,dx$$

$\vdots$

(iv) $\int \sec^n x \, dx$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$$

New parts $u = \sec x$ $v = \tan x$

$du = \sec x \tan x \, dx$ $dv = \sec^2 x \, dx$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x \, dx = \frac{1}{2}\{\sec x \tan x + \ln|\sec x + \tan x|\} + C$$