Separable Differential Equations

Notes, Examples, and Practice Exercises (w/Solutions)

Topics include natural logarithms, integrals, direct and inverse variation, Newton's Law of Cooling, and more.

Differential Equations

Example: Find the particular equation for $\frac{dy}{dx} = \frac{-x}{y}$ at (2, 0)

Step 1: Separate the x's and y's

cross multiply $y\,dy = -x\,dx$

Step 2: Integrate each side

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} + C = \frac{-x^2}{2} + C$$

Step 3: Rewrite AND substitute the given point (to find C)

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

@ (2, 0): $\frac{(2)^2}{2} + \frac{(0)^2}{2} = C$

$C = 2$

Step 4: Write final equation with C value

$$\frac{x^2}{2} + \frac{y^2}{2} = 2$$

The particular equation is $x^2 + y^2 = 4$

Example: Find the general solution for $x\frac{dy}{dx} + y = 2x^2 y$

Step 1: Separate the x's and y's

$$x\frac{dy}{dx} = 2x^2 y - y$$

$$x\frac{dy}{dx} = y(2x^2 - 1)$$

$$\frac{x}{(2x^2-1)}\frac{dy}{dx} = y$$

$$\frac{x}{(2x^2-1)}\frac{1}{dx} = y\frac{1}{dy}$$

$$\frac{(2x^2-1)}{x}\,dx = \frac{1}{y}\,dy$$

Step 2: Integrate both sides

$$\int \frac{(2x^2-1)}{x}\,dx = \int \frac{1}{y}\,dy$$

$$\int \frac{2x^2}{x} - \frac{1}{x}\,dx = \ln y + C$$

$$\int 2x - \frac{1}{x}\,dx = \ln y + C$$

$$x^2 - \ln x + C = \ln y + C$$

Step 3: Rewrite (solve for y)

$$x^2 - \ln x + C = \ln y$$ rewrite in exponential form

$$\log_e y = x^2 - \ln x + C$$

$$y = e^{x^2 - \ln x + C}$$ exponent laws

$$y = \frac{e^{x^2}}{e^{\ln x}} \cdot e^C$$ since e^C can be any constant, we can substitute any constant C

$$y = \frac{Ce^{x^2}}{x}$$

Example: Find a general solution for $x\frac{dy}{dx} + y = 0$

$$x\frac{dy}{dx} = -y$$

$$x\,dy = -y\,dx$$

$$\frac{1}{-y}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{-y}\,dy = \int \frac{1}{x}\,dx$$

$$-1\ln y = \ln x + C$$

$$\ln y = -\ln x - C$$

Again, since C is any constant, it can be negative or positive...

$$\ln y = \ln\frac{C}{x}$$

$$y = \frac{C}{x}$$

Separable Differential Equations: Using Separtion of Variables

Example: Find a general solution for the differential equation $\frac{dy}{dx} = e^{x - y}$

$e^{x - y}$ can be re-written as $\frac{e^x}{e^y}$ using basic exponent properties

$$\frac{dy}{dx} = \frac{e^x}{e^y}$$ If we cross-multiply, we separate the variables!

$$e^y\, dy = e^x\, dx$$ Then, integrate both sides

$$\int e^y\, dy = \int e^x\, dx$$

$$e^y + C = e^x + C$$ Combine the constants

$$e^y = e^x + C$$ Use logarithms to solve for y

$$\ln e^y = \ln(e^x + C)$$

$$y = \ln(e^x + C)$$

Example: Solve the differential equation by using separation of variables

$$\frac{dy}{dx} = (x + 2)(y + 5)$$

$$dy(1) = dx(x + 2)(y + 5)$$

Separate the variables

$$dy\,\frac{1}{(y + 5)} = dx(x + 2)$$

$$\ln(y + 5) + C = \frac{x^2}{2} + 2x + C$$

Since constants C are arbitrary, they can be combined...

$$\ln(y + 5) = \frac{x^2}{2} + 2x + C$$

Change logarithm to exponential form

$$e^{\frac{x^2}{2} + 2x + C} = y + 5$$

Logarithm 'power rule' (exponent C can be moved to coefficient of term)

$$C\,e^{\frac{x^2}{2} + 2x} = y + 5$$

$$y = C\,e^{\frac{x^2}{2} + 2x} - 5$$

Exponential Equations $y = ab^x$

Examples: $A = Pe^{rt}$ $A = A_0 (1 + \frac{r}{n})^{nt}$ $y = Ce^{kt}$

What is the rate of change?

$$\frac{dy}{dt} = Ce^{kt} \cdot k$$

$$= y \cdot k$$

$$\frac{dy}{dt} = ky$$

k is the constant of proportion

the rate of change varies directly with the output...

Direct Variation $y = kx$ where k is the constant of variation

Example: A model shows that $y = 14$ when $x = 7$.
If there is direct variation, then
if $x = 6/7$, what is y?

$y = kx$ model for direct variation

$14 = k(7)$ substitute x and y in order to find k

$k = 2$

$y = 2x$ find y

$y = 2(6/7)$

$y = 12/7$

Indirect Variation
("Inverse" Variation) $y = k\frac{1}{x}$ where k is the constant of variation

Example: A guy drives 4 hours to his vacation spot at 40 miles/hour.
If the travel time (between home and vacation spot) varies inversely to speed,
how long will the return trip home take at 60 miles/hour.

step 1: write inverse variation general equation $y = \frac{k}{x}$

what is the constant of variation k? the distance!

y = time of travel
x = speed of car

step 2: find constant of variation, using given information

$$4 \text{ hours} = \frac{k}{40 \text{ miles/hour}} \qquad k = 160 \text{ miles}$$

$$y = \frac{160 \text{ miles}}{x}$$

step 3: use equation to answer question $y = \frac{160 \text{ miles}}{60 \text{ m/h}} = 2\frac{2}{3}$ hours or 2 hours 40 minutes

Example: At a local preserve, the rate of change of the coyote population "P" is directly proportional to (650 − P), where the time is measured in t years. At the beginning, the first census showed a population of 300. Two years later, the population increased to 500. What is the projected population after 5 years?

"The population growth is directly proportional to (650 - P)

$$\frac{dP}{dt} = k(650 - P)$$

We have the rate of change...
Now, we want to find the population (growth) function..

$$dP = k(650 - P) \cdot dt$$

$$dP \cdot \frac{1}{(650 - P)} = k \cdot dt$$

$$\int \frac{1}{(650 - P)}\, dP = \int k\, dt$$

$$-\ln(650 - P) + C = kt + C$$

$$\ln(650 - P) = -kt - C$$

$$650 - P = e^{-kt - C}$$

$$650 - P = e^{-kt} \cdot e^{-C} \text{ — a constant...}$$

$$-P = Ce^{-kt} - 650$$

$$P = 650 - Ce^{-kt}$$

To find the specific solution....

$$P(t) = 650 - Ce^{-kt}$$

When $t = 0$, $P(t) = 300$

$$300 = 650 - Ce^{-k(0)}$$

$$C = 350$$

To find the growth (k)....

$$P(t) = 650 - 350e^{-kt}$$

When $t = 2$, $P(t) = 500$

$$500 = 650 - 350e^{-k(2)}$$

$$-150 = -350e^{-k(2)}$$

$$\frac{3}{7} = e^{-2k}$$

$$k = \frac{\ln(3/7)}{-2} = .4236$$

Finally, to find the number of coyotes at t = 5

$$P(t) = 650 - 350e^{-.4236t}$$

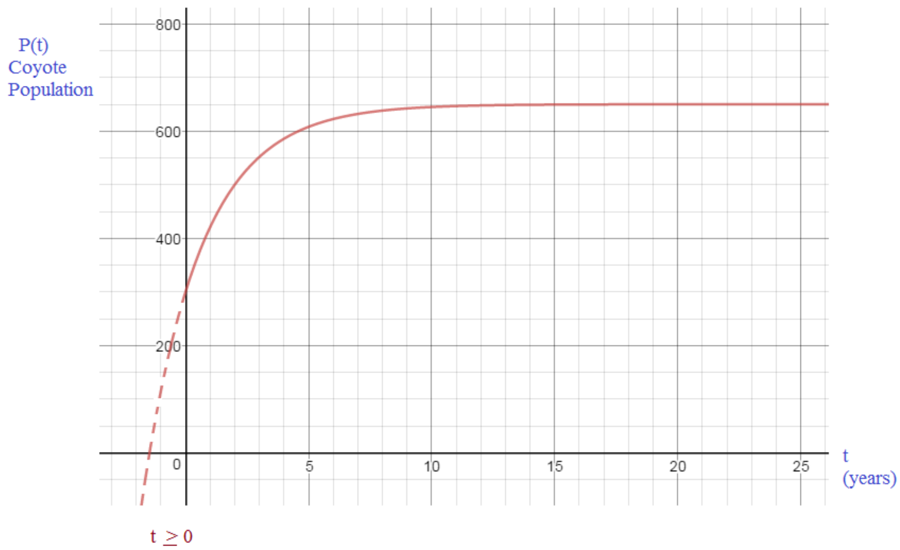
$$P(5) = 650 - 350e^{-.4236(5)}$$

$$= 607.9$$

Check: If t = 0... $P(0) = 650 - 350e^{-.4236(0)} = 300$

If t = 2.... $P(2) = 650 - 350e^{-.4236(2)} = 499.98$

Here's a graph...



Example: Newton's Law of Cooling states that the rate of change in temperature of an object is directly proportional to the difference between its temperature and the temperature of the medium.
An outdoor thermometer reading 85 degree F is brought into a 72 degree F room.
After 2 minutes, the thermometer reads 83 degrees.
When will the thermometer read 77 degrees? 75 degrees? 70 degrees?

$$\frac{dT_0}{dt} = k(T_0 - S)$$

T_0 = initial temperature (i.e. temperature of the object)

S = temperature of the medium (i.e. temperature of the Surroundings)

k = constant of proportion

$$\frac{dT}{dt} = k(T_0 - 72)$$

$$\frac{1}{(T_0 - 72)}\,dT = k\,dt$$

$$\int \frac{1}{(T_0 - 72)}\,dT = \int k\,dt$$

$$\ln(T_0 - 72) + C = kt + C$$

$$\ln(T_0 - 72) = kt + C$$

$$Ce^{kt} = T_0 - 72$$

$$T_0 = Ce^{kt} + 72$$

We know that the original temperature is 85 degrees when t = 0

$$85 = Ce^{k(0)} + 72 \qquad t = 0,\ T_0 = 85$$

$$C = 13$$

$$T(t) = 13e^{kt} + 72$$

2 minutes later, the temperature is 83 degrees...

$$83 = 13e^{k(2)} + 72$$

$$\frac{11}{13} = e^{2k}$$

$$\ln(11/13) = 2k$$

$$k = -.0835 \text{ (approx)}$$

$$T(t) = 13e^{-.0835t} + 72$$

If temperature cools to 77 degrees...

$$77 = 13e^{-.0835t} + 72$$

$$\frac{5}{13} = e^{-.0835t}$$

t = 11.44 minutes

If temperature is 75 degrees...

$$75 = 13e^{-.0835t} + 72$$

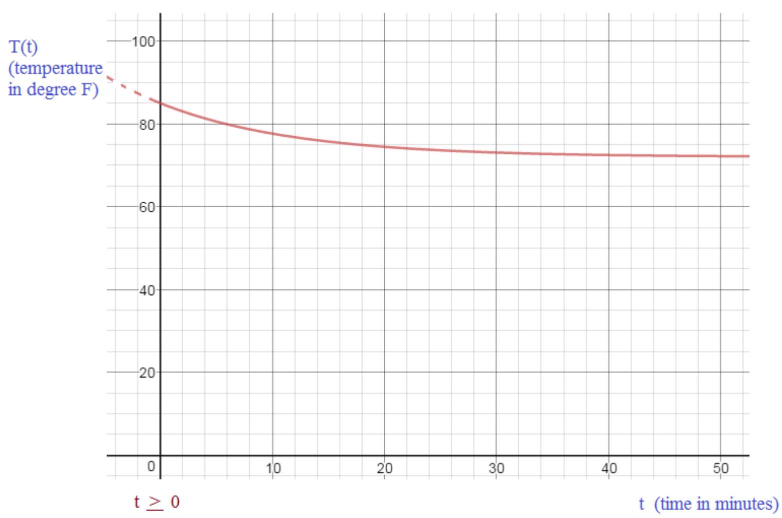
$$\frac{3}{13} = e^{-.0835t}$$

t = 17.56 minutes

To check: let t = 2 minutes

$$T(0) = 13e^{-.0835(0)} + 72 = 85$$

$$T(2) = 13e^{-.0835(2)} + 72 = 83$$



Can the temperature reach 70 degrees?

$$70 = 13e^{-.0835t} + 72$$

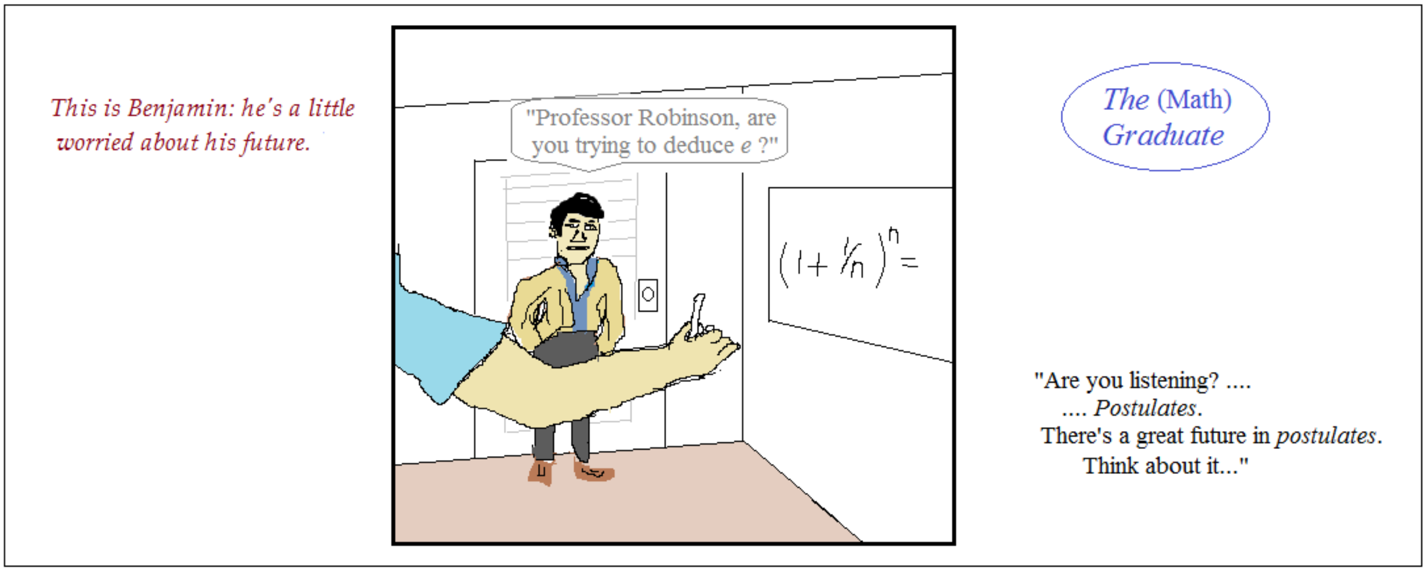
$$-2 = 13e^{-.0835t}$$

It's an exponential equation... The range is only positive numbers!

$$\frac{-2}{13} = e^{-.0835t}$$

$$\ln(-2/13) = -.0835t$$

NO SOLUTION



This sequel to *The Graduate* went straight to video.

LanceAF #32 5-13-12
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Practice Exercises-→

Separable Differential Equations: Exercises

1) Find the general solution: $\frac{dy}{dx} = 6x^2 + 3$

2) Find the particular solution: $\frac{dy}{dx} = 6x^2 + 3$

if $y = 30$ when $x = 1$

3) Find the general solution: $\frac{dy}{dx} = 2xy$

4) $\frac{dy}{dx} = 3x^2(y - 3)$ and $y = 12$ when $x = 0$.

Find the particular solution.

Separable Differential Equations: Applications

A) A calf that weighs 60 pounds at birth has a rate of change in weight with respect to time (t years) that is directly proportional to the difference between 1200 and its weight.

$$\frac{dw}{dt} = k(1200 - w)$$

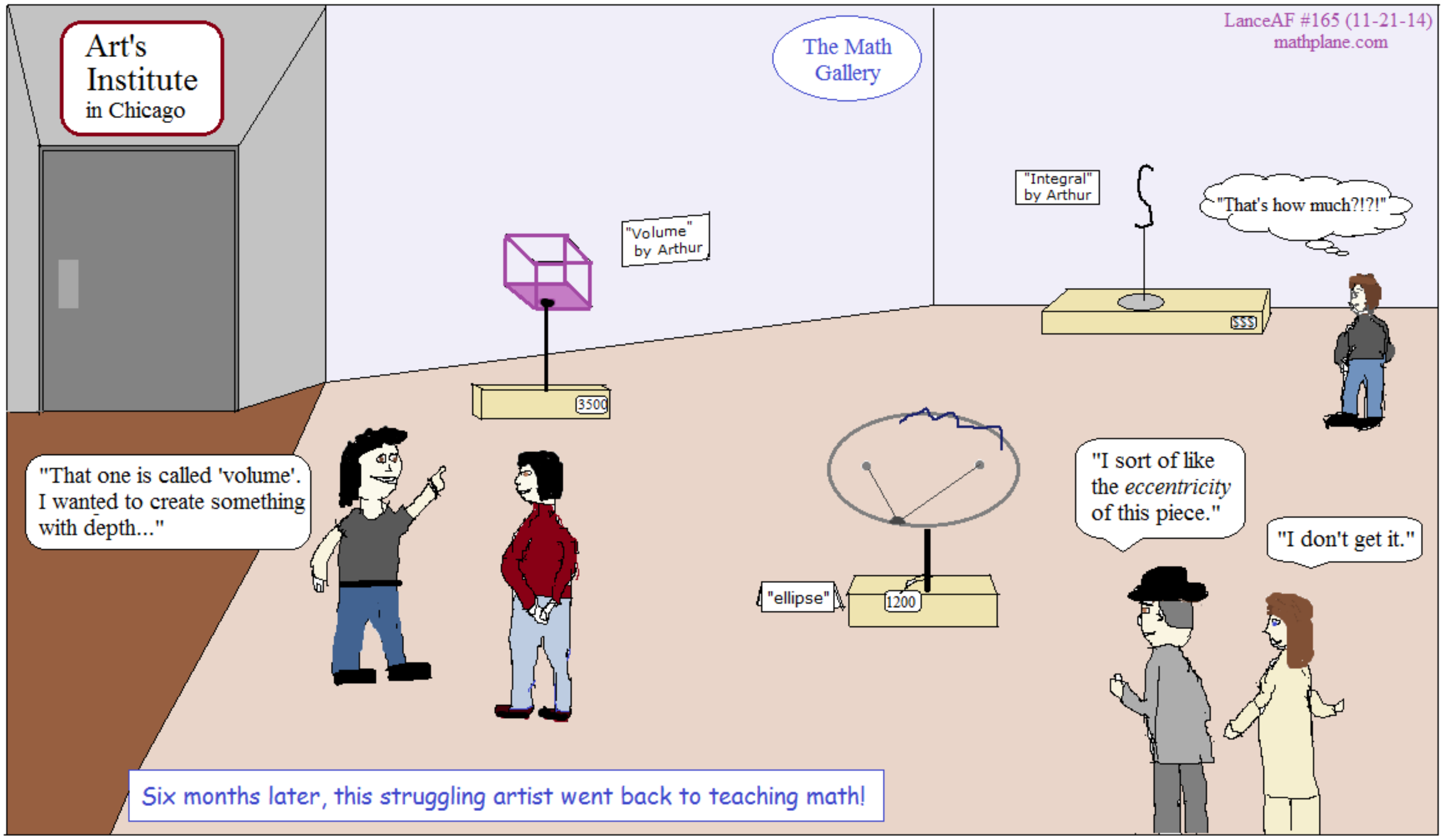
If k = .8, when will the calf's weight reach 800 pounds?

B) For my 30th birthday, my parents gave me $10,000 to save... I placed it in an account that pays 4% per year... Each month, I deposit $200 into the account...

a) If I received no interest, (just the original amount plus the $200 per month), how much will I have when I'm 50?

b) Using calculus create an equation that describes how much money you have at any given time..

c) How much will you have on your 40th birthday? 50th birthday?


Art's
Institute
in Chicago
The Math
Gallery
LanceAF #165 (11-21-14)
mathplane.com
"Volume"
by Arthur
3500
"Integral"
by Arthur
"That's how much?!?!"
$$$
"That one is called 'volume'. I wanted to create something with depth..."
"ellipse"
1200
"I sort of like the *eccentricity* of this piece."
"I don't get it."
Six months later, this struggling artist went back to teaching math!

SOLUTIONS-→

Separable Differential Equations: Exercises

SOLUTIONS

1) Find the general solution: $\frac{dy}{dx} = 6x^2 + 3$

$\frac{dy}{dx} = \frac{6x^2+3}{1}$ (cross multiply) to separate the variables

$1\,dy = 6x^2 + 3\;dx$ integrate

$\int 1\,dy = \int 6x^2 + 3\;dx$ simplify

$y + C = 2x^3 + 3x + C$

$y = 2x^3 + 3x + C$

(note: since the constants are arbitrary, they can be combined to make one constant C)

2) Find the particular solution: $\frac{dy}{dx} = 6x^2 + 3$ if $y = 30$ when $x = 1$

$\frac{dy}{dx} = \frac{6x^2+3}{1}$

$1\,dy = 6x^2 + 3\;dx$

$\int 1\,dy = \int 6x^2 + 3\;dx$

$y = 2x^3 + 3x + C$

substitute x and y: $30 = 2(1)^3 + 3(1) + C$

$30 = 5 + C$

$C = 25$

$y = 2x^3 + 3x + 25$

3) Find the general solution: $\frac{dy}{dx} = 2xy$

$\frac{dy}{dx} = \frac{2xy}{1}$ (cross multiply)

$1\,dy = 2xy\,dx$ separate the variables

$\frac{1}{y}\,dy = 2x\,dx$ integrate

$\int \frac{1}{y}\,dy = \int 2x\,dx$

$ln|y| + C = x^2 + C$

$ln|y| = x^2 + C$ (convert log function into exponential form)

$y = e^{x^2 + C}$

$y = e^{x^2} \cdot e^{C}$ (exponent laws)

$y = Ce^{x^2}$

(note: since C can be any number, presumably e^C can be any constant.)

4) $\frac{dy}{dx} = 3x^2(y-3)$ and $y = 12$ when $x = 0$.

Find the particular solution.

cross multiply $3x^2(y-3)\,dx = dy$

separate the variables (x and y) $3x^2\,dx = \frac{1}{(y-3)}\,dy$

integrate $\int 3x^2\,dx = \int \frac{1}{(y-3)}\,dy$

rearrange and combine the constants C $x^3 + C = \ln(y-3) + C$

convert to exponential form $\ln(y-3) = x^3 + C$

(exponent laws) $e^{x^3 + C} = y - 3$

Since C can be any constant e^C can be any number $y = e^{x^3} \cdot e^C + 3$

**So, we'll simplify and use C to be our constant, rather than e^C $y = Ce^{x^3} + 3$

To change the general solution to a specific solution, we use the given point:

(0, 12)

$12 = Ce^{0^3} + 3$

$12 = C(1) + 3$

$C = 9$

$y = 9e^{x^3} + 3$

Check: find derivative...

$\frac{dy}{dx} = 9e^{x^3} \cdot 3x^2 + 0$

we know that $9e^{x^3} = y - 3$

so, $\frac{dy}{dx} = (y-3) \cdot 3x^2$ ✓

A) A calf that weighs 60 pounds at birth has a rate of change in weight with respect to time (t years) that is directly proportional to the difference between 1200 and its weight.

Separable Differential Equations: Applications

SOLUTIONS

$$\frac{dw}{dt} = k(1200 - w)$$

If k = .8, when will the calf's weight reach 800 pounds?

$$\frac{1}{(1200 - w)}\,dw = k\,dt$$

$$\int \frac{1}{(1200 - w)}\,dw = \int k\,dt$$

$$-\ln(1200 - w) + C = kt + C$$

$$\ln(1200 - w) = -kt - C$$

$$1200 - w = e^{-kt - C}$$

$$1200 - w = e^{-kt} \cdot e^{-C}$$

$$w = -Ce^{-kt} + 1200$$

$k = .8$

$w(0) = 60$

$$w(t) = -Ce^{-kt} + 1200$$

$$w(0) = 60 = -Ce^{0} + 1200$$

$$C = 1140$$

$$w(t) = -1140e^{-.8t} + 1200$$

When will the calf reach 800 pounds?

$$800 = -1140e^{-.8(t)} + 1200$$

$$-400 = -1140e^{-.8(t)}$$

$$\frac{400}{1140} = e^{-.8(t)}$$

$t = 1.309$ years

B) For my 30th birthday, my parents gave me \$10,000 to save... I placed it in an account that pays 4% per year... Each month, I deposit \$200 into the account...

a) If I received no interest, (just the original amount plus the \$200 per month), how much will I have when I'm 50?

\$10,000 + \$200(240 months) = \$58,000

b) Using calculus create an equation that describes how much money you have at any given time..

F = amount in the Fund
t = time in months

$$\frac{dF}{dt} = 200 + \frac{.04}{12}F$$

$$dt = \frac{1}{200 + \frac{.04}{12}F}\,dF$$

$$\int dt = \int \frac{1}{200 + \frac{.04}{12}F}\,dF$$

$$t + C = \frac{12}{.04}\ln\left[200 + \frac{.04}{12}F\right]$$

$$\frac{.04}{12}t + C = \ln\left[200 + \frac{.04}{12}F\right]$$

$$e^{\frac{.04}{12}t + C} = \left[200 + \frac{.04}{12}F\right]$$

$$e^{\frac{.04}{12}t} \cdot e^{C} = 200 + \frac{.04}{12}F$$

$$C\,e^{\frac{.04}{12}t} = 200 + \frac{.04}{12}F$$

$$\frac{.04}{12}F = C\,e^{\frac{.04}{12}t} - 200$$

Since we have 10,000 dollars at the beginning t = 0
F = 10,000

$$\frac{.04}{12}(10{,}000) = Ce^{0} - 200$$

$$C = 233.333$$

$$F = \frac{12}{.04}\left[233.333e^{\frac{.04}{12}t} - 200\right]$$

c) How much will you have on your 40th birthday? 50th birthday?

10 years later ----> 120 months...

$$F(120) = \frac{12}{.04}\left[233.333e^{\frac{.04}{12}(120)} - 200\right] = \$44{,}427$$

$$F(240) = \frac{12}{.04}\left[233.333e^{\frac{.04}{12}(240)} - 200\right] = \$95{,}787$$

Note: if we test 1 month: t = 1

	\$10,000	initial fund
$\frac{.04}{12}(10{,}000) =$	\$33.33	interest
	\$200	contribution
	\$10,233.33	

then, using our formula:

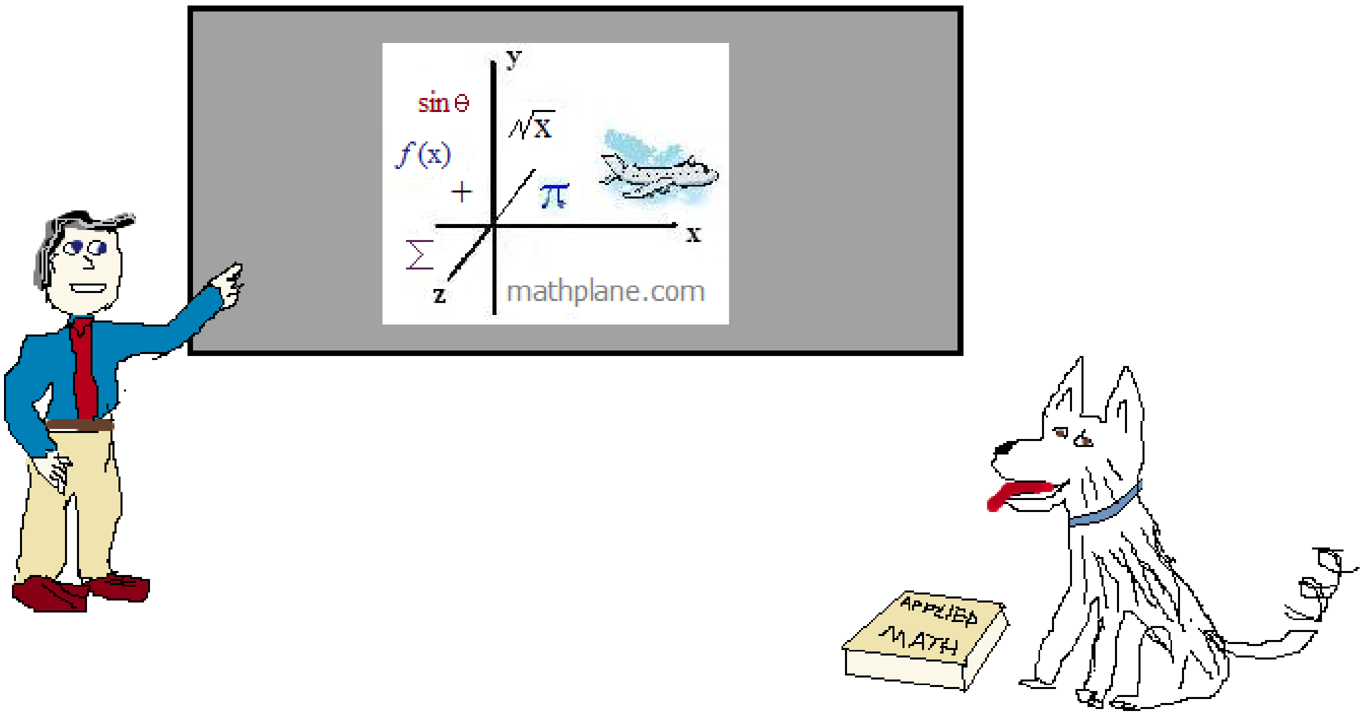
$$F(1) = \frac{12}{.04}\left[233.333e^{\frac{.04}{12}(1)} - 200\right] = \$10{,}233.72$$

**Note, our formula assumes the interest is applied daily, rather than monthly... So, there is a slight difference!

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions or requests, let us know.

Cheers.



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