Math 3331 - ODE's

Last class - we talked about differentials

that if $z = f(x,y)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

So if $z = x^2 y + x + 3y$

$$\frac{\partial z}{\partial x} = 2xy + 1, \quad \frac{\partial z}{\partial y} = x^2 + 3$$

& $dz = (2xy+1)dx + (x^2+3)dy$

If $(2xy+1)dx + (x^2+3)dy = 0$

$\Rightarrow \quad dz = 0 \Rightarrow z = \text{const}$

If we know that

$$(2xy+1)dx + (x^2+3)dy = 0$$

came from a differential then the sol^n of

would be $z = c$ (const)

Given $\frac{dy}{dx} = F(x, y)$

We will write this ODE in alternate form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

If this come (or comes) from a differential

then $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

Compare pieces

$$\frac{\partial z}{\partial x} = M(x, y), \quad \frac{\partial z}{\partial y} = N(x, y)$$

if z exists cross derivatives are the same

meaning

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ ← this is called the test of exactness

If this is true, z exists and

$z = f(x, y) = c$ is the sol^n of the DE

Ex $$\frac{dy}{dx} = \frac{3y - 3x^2y}{x^3 - 3x - 2y}$$

1st Put in alternate form

$$(x^3 - 3x - 2y)\,dy = (3y - 3x^2y)\,dx$$

or $$(3x^2y - 3y)\,dx + (x^3 - 3x - 2y)\,dy = 0$$

Identify that

$$M = 3x^2y - 3y, \quad N = x^3 - 3x - 2y$$

check for exactness $$\frac{\partial M}{\partial y} = 3x^2 - 3 \qquad \frac{\partial N}{\partial x} = 3x^2 - 3$$ they are the same so z exists

so $$\frac{\partial z}{\partial x} = M = 3x^2y - 3y$$

$$\frac{\partial z}{\partial y} = N = x^3 - 3x - 2y$$

Now integrate each including functions of integration

So $Z = x^3y - 3xy + A(y)$

$Z = x^3y - 3xy - y^2 + B(x)$

pick same pieces, pick $A(y)$ $B(x)$ to get a single Z

so $Z = x^3y - 3xy - y^2$

$$\boxed{\text{Sol}^n \quad x^3y - 3xy^2 - y^2 = c}$$

Ex 2 $\dfrac{dy}{dx} = \dfrac{xy^2 - \sin x \cos x}{y(1-x^2)}$, $y(0) = 2$

Alternate Form $y(1-x^2)dy + (\sin x \cos x - xy^2)dy = 0$

$M = \sin x \cos x - xy^2$ $\quad M_y = -2xy$

$N = y - x^2y$ $\quad N_x = -2xy$

> same so ODE is exact

so $Z_x = M = \sin x \cos x - xy^2$

$Z_y = N = y - x^2y$

integrate each

$$Z = \frac{\sin^2 x}{2} - \frac{x^2y^2}{2} + A(y)$$

$$Z = \frac{y^2}{2} - \frac{x^2y^2}{2} + B(x)$$

$$Z = \underbrace{-\frac{x^2y^2}{2}}_{\text{same}} + \underbrace{y^2/2}_{A(y)} + \underbrace{\frac{\sin^2 x}{2}}_{B(x)}$$

$$\text{Sol}^n \quad -\frac{x^2y^2}{2} + y^2/2 + \frac{\sin^2 x}{2} = c$$

we can absorb 2 into c

$$-x^2y^2 + y^2 + \sin^2 x = c$$

$$y(0)=2 \Rightarrow \quad -(0)^2(2)^2 + 2^2 + \sin^2(0) = c \Rightarrow c = 4$$

$$\boxed{\text{sol}^n \quad -x^2y^2 + y^2 + \sin^2 x = 4}$$

ex 3 $\quad \frac{dy}{dx} = \frac{2x-y}{x-2y}$ this ODE is homogeneous
It's also exact

$(2x-y)\,dx + (2y-x)\,dy = 0$

$M = 2x-y \quad N = 2y-x$

$M_y = -1 \quad N_x = -1$ ← so exact

$Z_x = M = 2x-y \Rightarrow Z = x^2 - xy + A(y)$

$Z_y = N = 2y-x \Rightarrow Z = y^2 - xy + B(x)$

$Z = \boxed{x^2 - xy + y^2 = C}$ sol^n

Pg. 15 Schaum's