Math 1497 Calc 2

A few trig preliminaries

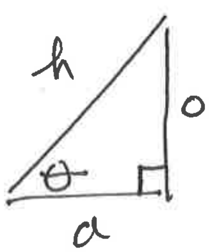
Trig. identities

$\sin^2 x + \cos^2 x = 1$, $\quad 1+\tan^2 x = \sec^2 x$

$\sin 2x = 2\sin x \cos x$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$

Right Angle Trig



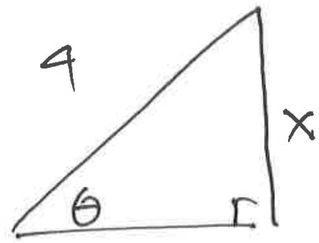
$\sin\theta = \dfrac{o}{h}$, $\quad \cos\theta = \dfrac{a}{h}$ $\quad \tan\theta = \dfrac{o}{a}$

$\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{h}{a}$ $\quad \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{h}{a}$ $\quad \cot\theta = \dfrac{1}{\tan\theta} = \dfrac{a}{o}$

Suppose I said that

$\sin\theta = \dfrac{x}{4}$ could I identify a triangle with this?

and then pick off all the other trig functions?



$$\sin\theta = \frac{o}{h} = \frac{x}{4}$$

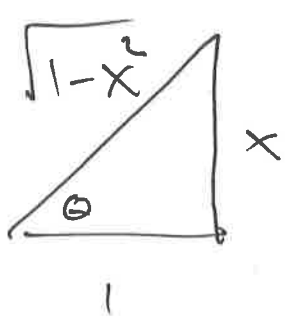
$\sqrt{4-x^2}$ ← Pythagoras

$$\text{so} \quad \cos\theta = \frac{a}{h} = \frac{\sqrt{4-x^2}}{4}$$

$$\tan\theta = \frac{o}{a} = \frac{x}{\sqrt{4-x^2}} \qquad \cot\theta = \frac{\sqrt{4-x^2}}{x}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{4}{\sqrt{4-x^2}} \qquad \csc\theta = \frac{4}{x}$$

if $\tan\theta = x = \dfrac{x}{1}$



$$\sin\theta = \frac{x}{\sqrt{1-x^2}}, \quad \cos\theta = \frac{1}{\sqrt{1-x^2}}$$

$$\sec\theta = \sqrt{1-x^2} \quad \text{etc}$$

we also need the trig functions

$\sin\theta$, $\cos\theta$, $\tan\theta$ in the 1^{st} Q

	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

& in reverse

divide

So for example

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\qquad$ $\tan\pi/3 = \sqrt{3}$

Also $\cos^{-1}\left(\frac{1}{2}\right) = \pi/3$ (go along $\cos\theta$ row to $\frac{1}{2}$ & go up to the angle)

\# 1 u sub

\# 2 parts

\# 3 trig $\int$

\# 4 trig sub ← today (& tomorrow)

Consider

$$\int \sqrt{a^2-x^2}\,dx$$

$$\int \sqrt{a^2+x^3}\,dx$$

$$\int \sqrt{x^2-a^3}\,dx$$

where a #

can we introduce a "trig. substitution" to simplify these $\int$'s?

a^2-x^2

if $x = a\sin\theta$

$$a^2-x^2 = a^2 - a^2\sin^2\theta$$

$$= a^2(1-\sin^2\theta)$$

$$= a^2\cos^2\theta$$

$$\sqrt{a^2-x^2} = a\cos\theta$$

$\underline{\underline{a^2+x^2}}$ if $x = a\tan\theta$

$$a^2+x^2 = a^2+a^2\tan^2\theta$$
$$= a^2(1+\tan^2\theta)$$
$$= a^2\sec^2\theta$$

$$\sqrt{a^2+x^2} = a\sec\theta$$

$\underline{\underline{x^2-a^2}}$ if $x = a\sec\theta$

$$x^2-a^2 = a^2\sec^2\theta - a^2$$
$$= a^2(\sec^2\theta - 1)$$
$$= a^2\tan^2\theta$$

$$\sqrt{x^2-a^2} = a\tan\theta$$

so it essence, we get rid of the $\sqrt{}$

So examples

$$\int \frac{x}{\sqrt{1-x^2}}\,dx \qquad u\text{-sub} \qquad u = 1-x^2$$

$du = -2x\,dx$ or $x\,dx = -\frac{du}{2}$

$$\Rightarrow \int \frac{-\frac{du}{2}}{\sqrt{u}} = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\,\frac{u^{1/2}}{1/2} + c$$

$$= -\sqrt{1-x^2} + c \quad \underline{\text{Ans}}$$

we try a trig sub

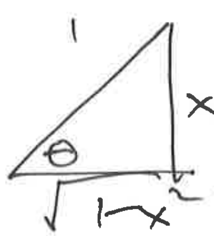
$x = \sin\theta \quad (a = 1)$

$dx = \cos\theta\, d\theta$

$1-x^2 = 1-\sin^2\theta = \cos^2\theta$

So $$\int \frac{x\,dx}{\sqrt{1-x^2}} = \int \frac{\sin\theta\cos\theta\,d\theta}{\sqrt{\cos^2\theta}} = \int \sin\theta\, d\theta$$

$$= -\cos\theta + c$$



$$-\frac{\sqrt{1-x^2}}{1} + c \quad \text{same}$$

$\sin\theta = \frac{x}{1}$

Ex $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$ ~~dx~~

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta}$

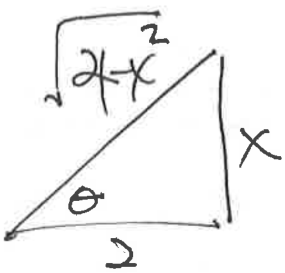
$= \sqrt{4(1+\tan^2\theta)}$

$= \sqrt{4\sec^2\theta}$

$$= \int \frac{\cancel{2}\sec^2\theta\, d\theta}{\cancel{2}\sec\theta}$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4-x^2}}{2} + \frac{x}{2}\right| + C$$



$\tan\theta = \frac{x}{2}$ $\quad \cos\theta = \frac{2}{\sqrt{4-x^2}}$ $\quad \sec\theta = \frac{\sqrt{4-x^2}}{2}$

Ex $\int \frac{dx}{x^2\sqrt{x^2-9}}$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

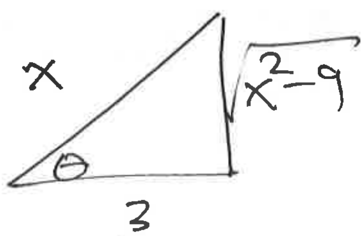
$\sqrt{x^2-9} = \sqrt{9\sec^2\theta-9}$

$= \sqrt{9(\sec^2\theta-1)}$

$= \sqrt{9\tan^2\theta} = 3\tan\theta$

$\int \frac{3\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{9\sec^{\cancel{2}}\theta\;3\cancel{\tan\theta}}$

$\frac{1}{9}\int \frac{d\theta}{\sec\theta} = \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$



$\sec\theta = \frac{x}{3}$ $\cos\theta = \frac{3}{x}$

So $\frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$

Ex $\displaystyle\int \frac{x^2}{(1-x^2)^{5/2}}\,dx$

$x = \sin\theta$ $\qquad 1-x^2 = 1-\sin^2\theta = \cos^2\theta$

$dx = \cos\theta\, d\theta$ $\qquad (1-x^2)^{5/2} = (\cos^2\theta)^{5/2} = \cos^5\theta$

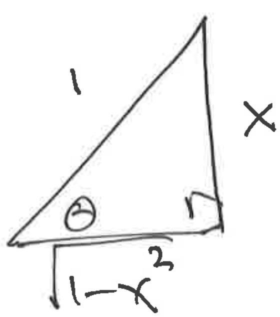
$$\int \frac{\sin^2\theta}{\cos^5\theta}\cos\theta\, d\theta = \int \frac{\sin^2\theta}{\cos^4\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\cos^2\theta}\,d\theta$$

$= \displaystyle\int \tan^2\theta \sec^2\theta\, d\theta$ ($\sec\theta$ even power)

$u = \tan\theta$ $\quad\Rightarrow\quad \displaystyle\int u^2\,du = \frac{u^3}{3} + C$

$du = \sec^2\theta\, d\theta$

$$= \frac{1}{3}\tan^3\theta + C$$



$\sin\theta = \frac{x}{1}$ so $\tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

Ans $\quad \dfrac{1}{3}\left(\dfrac{x}{\sqrt{1-x^2}}\right)^3 + C$ or $\dfrac{1}{3}\dfrac{x^3}{(1-x^2)^{3/2}} + C$

Trig Sub w/ limits

Ex $\displaystyle\int_0^{1/2} \frac{dx}{(1-x^2)^{3/2}}$

so $x = \sin\theta$ $\quad$ $x=0$ $\sin\theta = 0$ $\quad \theta = 0$

$dx = \cos\theta \, d\theta$ $\quad$ $x = \frac{1}{2}$ $\sin\theta = \frac{1}{2} \Rightarrow \theta = \pi/6$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$$

so $(1-x^2)^{3/2} = \cos^3\theta$

$$\int_0^{\pi/6} \frac{\cos\theta \, d\theta}{\cos^3\theta} = \int_0^{\pi/6} \sec^2\theta \, d\theta = \tan\theta \Big|_0^{\pi/6}$$

$$= \tan\frac{\pi}{6} - \tan 0 = \frac{1}{\sqrt{3}}$$

Ex $\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta$

$x^2-4 = 4\sec^2\theta - 4$

$= 4(\sec^2\theta - 1)$

$= 4\tan^2\theta$

$\int \frac{2\tan\theta\,\cancel{2}\cancel{\sec\theta}\tan\theta\,d\theta}{\cancel{2}\cancel{\sec\theta}}$

limits

$x=2 \quad 2\sec\theta = 2$

$\sec\theta = 1$

$\cos\theta = 1$

$\theta = 0$

$x=4 \quad 2\sec\theta = 4$

$\sec\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \pi/3$

$2\int_0^{\pi/3} \tan^2\theta\,d\theta$

$2\int_0^{\pi/3} (\sec^2\theta - 1)\,d\theta$

$2\left(\tan\theta - \theta \Big|_0^{\pi/3}\right) = 2\left(\left(\tan\pi/3 - \pi/3\right) - 0\right)$

$2\left(\sqrt{3} - \pi/3\right)$

Ex $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+1}}$

$x = \tan\theta \qquad x^2+1 = \tan^2\theta+1$

$dx = \sec^2\theta\, d\theta \qquad = \sec^2\theta$

<u>limits</u> $x=1 \quad \tan\theta = 1 \quad \theta = \pi/4$

$x = \sqrt{3} \quad \tan\theta = \sqrt{3} \quad \theta = \pi/3$

$$\int_{\pi/4}^{\pi/3} \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta} = \int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta} = \int_{\pi/4}^{\pi/3} \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta}\, d\theta = \int_{\pi/4}^{\pi/3} \cot\theta\csc\theta\, d\theta$$

$$= -\csc\theta \Big|_{\pi/4}^{\pi/3} = -\left(\csc\pi/3 - \csc\pi/4\right)$$

$$= -\left(\frac{1}{\sin\pi/3} - \frac{1}{\sin\pi/4}\right) = -\left(\frac{2}{\sqrt{3}} - \frac{2}{\sqrt{2}}\right)$$