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Optimus prime: paraphrasing prime number talk

Jonathan Tallant

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Abstract Baker (Mind 114:223–238, 2005; Brit J Philos Sci 60:611–633, 2009) has recently defended what he calls the “enhanced” version of the indispensability argument for mathematical Platonism. In this paper I demonstrate that the nominalist can respond to Baker’s argument. First, I outline Baker’s argument in more detail before providing a nominalistically acceptable paraphrase of prime-number talk. Second, I argue that, for the nominalist, mathematical language is used to express physical facts about the world. In endorsing this line I follow moves made by Saatsi (Brit J Philos Sci 62(1):143–154, 2011). But, unlike Saatsi, I go on to argue that the nominalist requires a paraphrase of prime-number talk, for otherwise we lack an account of what that ‘physical fact’ is in the case of mathematics that seemingly makes reference to prime numbers.

Keywords Mathematical Platonism · Nominalism · Indispensability argument · Enhanced indispensability argument · Baker · Saatsi

1 Introduction

Baker (2005, 2009) “enhanced” version of the indispensability argument for mathematical Platonism, runs thus:

- (1) We ought rationally to believe in the existence of any entity that plays an indispensable explanatory role in our best scientific theories.

Optimus prime, literally, “Best Prime”.

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- (2) Mathematical objects play an indispensable explanatory role in science.
- (3) Hence, we ought rationally to believe in the existence of mathematical objects.

The difference between Baker’s “enhanced” indispensability argument and the typical formulations of the argument consists in Baker’s demand that we ask for cases where mathematical entities play an *indispensable explanatory role* in our best theories. Typical versions of the argument demand only that we have ontological commitment to entities that are indispensable to our theories.¹

Baker thinks that there is at least one instance of a mathematical object playing such an explanatory role. The *only* case that Baker offers is that of the life cycle of the periodical cicada, “an insect whose two North American subspecies spend 13 years and 17 years, respectively, underground in larval form before emerging briefly as adults” (Baker 2009, p. 614).

There is a question in the offing: why are these life cycles prime? The answer is that prime cycles minimise overlap with other periodical organisms. Thus, the thought is that the ‘primeness’ of the number plays some explanatory role. Here is Baker’s (2009, p. 614) formalisation of the argument:

- (4) Having a life-cycle period that minimizes intersection with other (nearby/lower) periods is evolutionarily advantageous (biological law).
- (5) Prime periods minimize intersection (compared to non-prime periods). (number theoretic theorem).
- (6) Hence organisms with periodic life cycles are likely to evolve periods that are prime (‘mixed’ biological/mathematical law).
- (7) Cicadas in ecosystem-type E are limited by biological constraints to periods from 14 to 18 years. (ecological constraint).
- (8) Hence cicadas in ecosystem-type E are likely to evolve 17-year periods.

A crucial claim that Baker makes is that primeness cannot be paraphrased away (2009, p. 619) in a fashion acceptable to a mathematical nominalist. After all, if “primeness” is what is doing the explanatory work, and “primeness” can be paraphrased, then Baker would have failed to show that there is a mathematical entity—the property of primeness—that’s doing any explanatory work.

Baker (2009, p. 619) looks to defend the claim that primeness cannot be given a nominalistic paraphrase in the following passage.

As has been argued above, the fact for which biologists sought an explanation involves the notion of primeness. In addition, claims involving primeness such as

- (10) The number of F’s is prime

cannot be paraphrased into first-order logic so as to eliminate any mention of primeness. The reason, in a nutshell, is that there are an infinite number of ways for a number to be prime; hence, any paraphrase of (10) would have to involve an infinite disjunction of the form, ‘X has life-cycle length 2, or length 3, or length 5; or length 7 or ...’

¹ E.g. Colyvan (2001, p. 11).

If this were correct, then it would be bad news for the mathematical nominalist. I think that there is a paraphrase to hand that eliminates the need to think that there are mathematical entities. In Sect. 6 I will also argue that the disjunction need not be infinite in length.

To make one thing clear, my target is not to persuade the nominalist, or anyone else, that this is what they *should* say. Rather, my more limited goal is to offer the nominalist about mathematics an acceptable general paraphrase of prime number-talk that will serve to counter Baker's claim. The strategy deployed to support this argument is as follows. I begin with a definition of what is required for a number to be prime. I then offer a nominalistically acceptable paraphrase of certain key mathematical terms that are used in the definition of primeness and then, using these, show how an acceptable paraphrase of prime-number talk can be constructed.

2 The paraphrase

Let us begin with a cursory definition of what it is for a number to be regarded as prime.

(PN) A natural number is prime iff it has exactly two distinct natural number divisors; 1 and itself

What we must do, then, is find some way of articulating PN in a way that does not commit us to talk of numbers. It is worth pausing a moment to remind ourselves how the nominalist will, typically, understand talk about numbers.

Typically, nominalists will look to paraphrase natural number-talk using first order logic plus identity. Thus, to say that there are 'three Fs' we would say that: $\exists x \exists y \exists z Fx \ \& \ Fy \ \& \ Fz \ \& \ \neg(x = y) \ \& \ \neg(x = z) \ \& \ \neg(y = z) \ \& \ \forall t Ft \rightarrow t = x \vee t = y \vee t = z$. So far, so familiar. This familiar nominalist treatment of numbers is one that I will look to extend and defend in this paper.

What this typical paraphrase makes clear is that, according to the nominalist, much of our typical number-talk is in fact *about* particulars. To say that there is "one F", is to make a commitment to there *being an F*. To say that there are "two Fs" is to make a commitment to *there being an F and being another F*—and so on. This is salient when we turn our attention to primes. It *appears* that when we say something of the form "*n* is a prime number" we are attributing a property to a number: the property of primeness to the number *n*. Given that the aforementioned nominalist tradition treats number-talk as talk *about particulars*, so to make the claim that a number "is prime" is to make a claim about some particulars. (Whilst nothing that has been said *entails* that this is the case, it seems like a reasonable inference.) So in what follows we must be clear that, for the nominalist, "being prime" is not a property of numbers *per se*, but is a claim about a collection of particulars. With that background in place, let us now move on to consider the criteria, the satisfaction of which will suffice for primeness.

The first point is this. 1 is not a prime number, so our paraphrase must be such as to rule out a lone F "being prime." We thus require that when we attribute primeness, if we are doing so correctly, we are making a claim about *particulars* (plural).

The second point concerns the notion of a divisor. We saw, in PN, that the notion of a divisor plays a key role in the definition of primeness. How are we to understand the notion of a divisor if we're nominalists? It will facilitate what follows to have to hand an account of equinumerosity.²

The following, Fregean, analysis gives us a sense of the desired result—though it will require modification. *F* and *G* are equinumerous just in case there is a relation *R* such that: (1) every *F* is *R*-related to a unique *G*, and (2) every *G* is such that there is a unique *F* which is *R*-related to it. More formally, we might say that *F* is equinumerous to *G* iff (Zalta 2010),

$$\exists R [\forall x (Fx \rightarrow \exists! y (Gy \ \& \ Rxy)) \ \& \ \forall x (Fx \rightarrow \exists! y (Gy \ \& \ Ryx))]$$

The trouble we face with this Fregean analysis, as good nominalists, is one of quantifying over the relation *R*. I assume that the mathematical nominalist does not want to endorse the existence of 'relations' as entities. In place of this, I propose a different analysis.

A natural way to think of the equinumerosity of the *F*s and *G*s is to think of the *F*s and *G*s being such that we can pair them up, such that each unique *F* is paired with a *G*, and vice versa. In lieu of this 'pairing up' operation, I propose the following analysis in terms of fusions:

EQUINUMEROSITY:

The *F*s and *G*s are equinumerous iff the fusion of the *F*s and *G*s is overlapped by fusions, the *H*s, such that each *H*³:

- (i) has one and only one *F* as a part,
- (ii) has one and only one *G* as a part,
- (iii) Every *F* and every *G* is a part of one, and only one, *H*⁴

The *H*s specified are the 'pairs' of an *F* and a *G*. Since each *H* includes precisely one *F* and one *G*, and every *F* and every *G* is a part of some *H*, so we ensure that the *F*s are equinumerous to the *G*s.

Using this, we can then specify the notion of a divisor. As was made clear above, talk of numbers greater than one is, properly construed, talk about collections of individuals. That being the case, to say that 'x is a divisor of y', where x and y are numbers greater than one will require us to talk about one collection being a divisor of another collection. We must say that, properly speaking 'the *F*s collectively are a divisor of the *G*s'. We must, therefore, reconstrue natural language talk of 'x being a divisor of y' as 'the *x*s being a divisor of the *y*s'. Given nominalism, and that we are treating concrete objects as the surrogates for our number talk, this seems both natural and unsurprising. We may then complete the analysis.

² Informally, one number is a divisor of another iff it divides the second number without remainder.

³ I assume Universalism such that for any disjoint *x*s, there is a fusion of the *x*s. To say that the *F*s overlap the *G*s is to say that every *F* is a part of a *G* and that every *G* has only *F*s (or other fusions constructed solely from *F*s, or entities that themselves compose *F*s) as parts.

⁴ It follows from this analysis that equinumerosity is reflexive, symmetric and transitive.

Let us think, momentarily, about the way in which division is often (crudely) thought of. Indeed, let us think about the way in which we might teach a young child about division. An oft-deployed idiom is that of ‘goes into’: we say that 2 divides 4 because ‘2 goes into 4, twice’. We can apply this thought to particulars—the Fs. Suppose that we had a collection of Fs and a collection of Gs. Are the Gs a divisor of the Fs? It will depend upon whether or not the Gs ‘go into’ the Fs.

To see whether they do, let us ask whether or not there are fusions of Fs, call them the Js, that overlap the fusion of all of the Fs, such that there are as many Fs in each J as there are Gs. If that is the case, then the Gs can be said to ‘go into’ the Fs, because the Js do, and there are as many Fs in each J as there are Gs. Formally, then:

DIVISOR:

The Gs are a divisor of the Fs iff there is a fusion of the Fs; the fusion of the Fs is overlapped by fusions, the Js, such that the Fs in each J are equinumerous with the Fs in each other J; the Fs in each J are equinumerous to the Gs.

The analyses of equinumerosity and what it is to be a divisor then permit an analysis of primes. We learned, above, that a number is prime iff it has only two divisors, one and itself. We have also learned how to think of divisors, and we have reminded ourselves that number talk is, properly understood, talk about particulars.

The forgoing ideas then suggest that we ought to say that ‘the number of Fs is prime’ is true iff either, the only divisors of the Fs are a lone G (to stand in for the claim that ‘1’ is a divisor of a prime), and a collection of Gs that are equinumerous to the Fs (to stand in for the claim that primes are divisible by themselves). Thus,

PRIME:

‘The number of Fs is prime’ is true iff there are Fs (plural), and any divisor of the Fs is equinumerous to either *an* F, or to all of the Fs.

This concludes the introduction to the paraphrase. It is easy to see how the above will apply to Baker’s case. We simply require that the lifecycle of the periodic cicada is such that it matches the above specification.

(Thus, there are years (the lifecycle of the cicada) and any divisor of the years is either equinumerous to *a* year, or equinumerous to all of the years; since the lifecycle of the cicada is 17 years, so any divisor would have to include 17 elements.)

3 Explanation and the lemmas

However, although this may give us a paraphrase of prime number talk, it is not yet clear that the nominalist has met Baker’s challenge.

There are two putative explanations for the lifecycle of the periodic cicada that Baker (2005, pp. 230–231) discusses.⁵ Baker claims (2005, pp. 231–232) that what is common to both explanations is that they rely explicitly upon a number theoretic

⁵ I don’t discuss the details of these since the contents of the explanations are not salient to the current paper.

theorem: prime periods minimise intersection (compared to non-prime periods). Thus, it is Baker's contention that both of the putative explanations of the lifecycle of the cicada can be explained, in part, by reference to number theory. If that's right, then *merely* that we can paraphrase "the number of Fs is prime" does not fully discharge our explanatory duty. We have yet to paraphrase the relevant portions of number theory that Baker claims are doing explanatory work.

Baker's explicit contention (2005, p. 232) is that it is the lemmas cited in his argument that serve to underpin the explanation. In order to offer a satisfactory riposte to the enhanced indispensability argument, I now turn to offering a complete and satisfactory paraphrase of these lemmas.

The completion of the project requires two pieces of terminology, and an explanation of these: "lowest common multiple" (lcm), and "co-prime".

3.1 Background to the lemmas

Here is Baker's (2005, p. 231) account of lcm. 'The lcm of two natural numbers, m and n , is the smallest number into which both m and n divide exactly.' To borrow Baker's example, the lcm of 4 and 10 is 20.

Some of the preliminaries to this will be obvious. We will require that the concept of a plurality—the lcm itself—and we will require it to have two divisors—the two natural numbers, m and n adverted to above. In place of m and n , and in accordance with the strategy adopted here, I propose we think of Fs and of Gs.

Things require a little more imagination when it comes to specifying that the plurality is the *lowest* common multiple. Suppose that we have Ls, the plurality that is our putative lcm, that the Fs that are a divisor of the Ls, and that the Gs that are another divisor of the Ls.

We must then avail ourselves of the notion of 'smaller than', as it applies to pluralities. Suppose that we have some Fs and some Gs. If the Fs are fewer than the Gs, then the Fs will be equinumerous to some, but not all, of the Gs. Another way to put this is to say Fs are equinumerous to the Gs that are a proper part of the fusion of all of the Gs. We may then make use of this idea to talk about the lcm.

If the Ls (that are standing in for the lcm) are the smallest number of Ls that have both the Fs as a divisor and the Gs as a divisor, then any other plurality of particulars that has both the Fs and the Gs as a divisor is either equinumerous to the Ls or has a proper part that is equinumerous to the Ls. Putting this together with the above then permits an account of the lcm

LCM:

The Ls are a lcm of the Fs and the Gs iff there are Fs and there are Gs and there are Ls (to stand in for the lcm); the Fs are a divisor of the Ls, and the Gs are a divisor of the Ls; and for any Ps of which both the Fs and the Gs are a divisor, the Ps are either equinumerous to the Ls, or the fusion of the Ps has a proper part, consisting of Ps, such that the Ps in the proper part of the fusion of the Ps are equinumerous to the Ls.

We then need the concept of a co-prime. To say of two numbers that they are ‘co-prime’ is to say that they have no common factors other than 1; e.g. 9 and 10 satisfy these conditions. To then say of two pluralities of particulars that they are co-prime is, roughly, to say that they do not have a common divisor, other than where that divisor is only a single entity. More formally:

CO-PRIME:

the Fs are co-prime with the Qs iff a lone G is the only divisor of both the Fs and the Qs

3.2 To the lemmas

The idea that ‘ m and n are maximal’ is intended to convey is roughly that, relative to other numbers of similar value, the points at which m and n intersect are higher than other numbers of similar value. In our case, the claim under consideration is that co-primes are maximal with respect to other numbers of similar value. For instance, compare 6 and 12 with 5 and 11. 6 and 12 are not co-prime; both are divisible by 2 and 3. The lcm of 6 and 12 is merely 12. In contrast, the lcm of 5 and 11 (which are co-prime) is 55. Thus, as this brief example reveals, co-primes tend to maximise points of intersection.

This gives us the first Lemma.

Lemma 1 *the lowest common multiple of m and n is maximal if and only if m and n are co-prime*

Here is the nominalist account of Lemma 1.

Nominalist Lemma 1 (NL1): the lcm of the Fs and the Gs, *the Ls*, is maximal if and only if the Fs and the Gs are co-prime

It is a simple fact about any given plurality of particulars that a plurality is larger where it has two divisors, the Fs and the Gs, that are co-prime, than where it does not. What we want, of course, is an explanation of the salience of *primeness*. NL1 simply secures the result that co-primeness ensures a relatively large lcm. Thus, let us now turn to the second lemma.

Lemma 2 *a number, m , is co-prime with each number $n < 2m$, if and only if m is Prime*

The part of lemma 2 that we have not yet encountered (and that will prove complex to paraphrase) is that ‘ m is co-prime with each number $n < 2m$ ’.

The strategy I deploy here, is this. I begin by sketching an account of ratio-talk, such as to enable us to say that the ratio of Fs to Gs is 1:2. We will then say that $n < 2m$ if and only if the Fs are equinumerous to a proper part of the fusion of the Gs. Having already established that the ratio of Fs to Gs is 1:2, this will give us the result that there are at least twice as many Gs as there are Fs. Let us then move to consider ratio-talk.

To establish the required ratio of 1:2, we must simply say that:

A RATIO OF 1:2:

There is a fusion of the Fs and the Gs that are overlapped by fusions, the Hs, such that each H has:

- (i) one and only one F as a part
- (ii) two and only two Gs as a part
- (iii) Every F and every G is a part of one and only one H.

This establishes that there are twice as many Gs as Fs.

Returning our attention to the issue of the second lemma, we are now in a position to understand the claim that $n < 2m$. In place of n , we have the Fs. To say that n is *less than* $2m$, we must say that the Fs are equinumerous to the Us, where the Us themselves are equinumerous to the Gs that are a *proper part* of the fusion of all of the Gs. The nominalist may, thus, assert the second lemma without any difficulty; each part of the lemma can be paraphrased. This completes our explanatory duty.

4 Problems and replies

First, can we really talk about “collections of years”? After all, this does seem to pre-suppose that ‘years’ are things over which we may quantify. If we cannot so quantify, then the view proposed here is obviously in trouble.

I think we can. Assume, for ease of exposition, an eternalist framework (according to which the past, present and future all exist). Further, let us borrow from Field (1980, p. 62) and suppose that we may quantify over space-time points. A year, on such a framework, is nothing more than a particular temporal distance—a particular number of space-time points, in a given direction. On the further assumption that these distances are parts of greater distances (e.g. that a 1 year period is a part of a 5-year period), then we can talk of collections of years.

Second, there is a question in the vicinity concerning whether or not the notion of “yearhood” is sufficiently ontologically robust. In the context of the cicada explanation, the distance cannot be completely precise. For example, a 17-year cicada subspecies will emerge on a particular date but may not emerge on that same date 17 years later (this will depend on factors such as soil temperature and rainfall in the given year). Hence the period here may well not be a group (or fusion) of 17 “temporal distances” of exactly the same length.

However, we *could* make precisely the same point against Baker. It is not the case, after all, that Baker thinks that cicada have *exactly* prime lifecycles. To be clear, then, the proper explanatory task upon which we are engaged here is not “why do North American cicada have precisely 17 year life-cycles?”, because they do not. There will be some minimal deviation from the 17 year period. What is in need of explanation is that the life-cycle of the North American cicada closely approximates a 17 year periodic lifecycle. That question is readily answered simply via the above demonstration that a 17 year periodic lifecycle minimises intersection. Thus, by approximating said 17 year periodic lifecycle closely, the cicada is able to minimise intersection.

Finally, the complaint may be that some proof of one of the lemmas makes use of a mathematical device that, itself, cannot be paraphrased. If it could be shown that such

a mathematical proof is integral to the explanation of the life-cycle of the cicada, then I would concede that there is a case to answer.

However, no such case has yet been forthcoming. Baker's claim is that the lemmas themselves serve to underpin the mathematical explanation of the life-cycle of the cicada, and these lemmas have been paraphrased. If there is more to be said about the proofs, then Baker must be forthcoming with a response.

5 Under ambitious?

There is a further concern, however, that without a *complete* paraphrase for *all* mathematical discourse, I'm doing nothing substantial. There is nothing to stop the Platonist simply finding *another* case where mathematical entities play an explanatory role. In that case, nominalism would be on the back foot once more. Now, as is clear, there are problems for the program that I do not consider here (see, for instance, Melia (1998)).

Two remarks, then, by way of response. First, it would be highly uncharitable to object to an attempt to paraphrase *some* of our mathematical discourse that it does not give us a paraphrase for *all* of our mathematical discourse. As I have made plain from the outset, my primary task in this paper is to provide a paraphrase of prime-number talk. I have done that. And, although I will offer some further paraphrases in a moment (*i*, and, π), I concede that the program is not complete. But given the dialectic, it is not clear how serious a problem this is. Baker's case is intended to tip the argument against those who claim that mathematical explanations do not exist by showing that we have *an* instance of a genuine mathematical explanation (see, e.g. (2005, p. 229)). My claim is that we do not have such a case, because we have a paraphrase of the requisite mathematical language and, as we shall see in more detail in Sect. 6, that this permits us to offer a genuine explanation of the phenomena involved. The onus is, therefore, back on the Platonist to persuade us to believe in abstract objects.

Second, I do think that there are applications for this kind of strategy that suggest that it takes us beyond the level of 'mere response' to Baker. The account of equinumerosity that was given above is a useful tool in the nominalist armoury and, with a little careful analysis, can be developed of be of further use.

As was noted above, we can use the account of equinumerosity that was specified to work out how to deal with ratio-talk. To see what sorts of problems this might solve, consider a problem due to Melia (1995, p. 226):

Consider, for example, the sentence 'the average star has 2.4 planets'. Perhaps the implications for the concrete world that this sentence has are indeed correct—suppose that there are precisely twenty four zillion orbiting planets and ten zillion stars. But whilst we may have very good evidence that the average star has 2.4 planets, we may not have any evidence at all that there are precisely twenty four zillion planets and ten zillion stars. And our chances of counting up all the stars and planets are, to say the least, slim. Indeed, I think that our ignorance in this case is ineliminable, and so the manoeuvre considered above is not open to us. When our ignorance is ineliminable, the programme simply cannot be carried through.

As Melia then goes on to point out:

If we omit [from our paraphrase] ‘the number of orbiting planets divided by the number of stars equals 2.4’ we fail to represent something true about how many stars and planets there are.

With a little modification we can carry through the nominalist programme and thus say that ‘the number of orbiting planets divided by the number of stars equals 2.4’.

We can then explain Melia’s case: we must say that there some planets, the Fs; there are some stars, the Gs; the Hs overlap the fusion of the stars and the planets such that each H has five and only five stars as a part and 12 and only 12 planets as a part, and every star and every planet is a part of one and only one H.⁶

All of which raises the question of how far these paraphrases can go. In this section we are considering the propriety of the expansiveness of the project. Can *all* scientifically explanatory mathematics be given a mereological treatment?

Answering the question is difficult, because there is no hard-and-fast agreement as to what constitutes ‘scientifically explanatory mathematics’. What I’ve done here is sketched some of the tools that are available to the nominalist. I’m optimistic about the expansion of the project, but it remains to be seen how far it can be taken. One obvious weakness in the forgoing is that I have said nothing, here, about the irrational or complex numbers. Let us begin with the complex number, $i : \sqrt{-1}$. Obviously, specifying a nominalist treatment for this appears complex.

The ubiquity of i is my chief concern; it appears in so many mathematical formulae that it seems highly likely that at least some explanatory role can be attributed to it. This therefore threatens the current nominalist proposal.

Following the Caley–Dixon construction, we can write a complex number as a pair of real numbers. Since we already know how to treat number-talk—as referring to particulars, or collections of particulars—so the explanation proposed ought not to bring with it any ontological complications.

The following is our rule for adding two complex numbers

$$(a, b) + (c, d) = (a + c, b + d)$$

We also have a rule for multiplying two complex numbers:

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

where the second component of the complex number is ‘0’, the complex number is associated with a real number: the complex number $(a, 0)$ can be said to be equivalent to the real number a . The ‘complex number’ (a, b) is, depending upon the values in question given the nominalist treatment: there are some Fs, there are some Gs.⁷

Since we are after i , what we want is a complex number (that is, a pair of real numbers) that, given the above laws of arithmetic, are such that squaring them generates

⁶ As elsewhere, ‘5’ and ‘12’ could be given nominalist treatments.

⁷ As I understand it, the Caley–Dixon construction can be extended in similar fashion to deal with Quaternions and Octonions. See, e.g., Dixon (1919)

the pair $(-1, 0)$. The pair that achieves this is $(1, 0)$.⁸ Thus we may say that the pair of real numbers $(1, 0)$ expresses the content of i . We thus have an explanation of i that makes no recourse to anything ‘imaginary’ or abstract.

The irrational numbers do not pose a serious problem. An irrational number is one that cannot be represented as a fraction using the rational numbers, nor can it be represented as a terminating or repeating series of decimals. We may grant, then, that irrationals such as e , π , and the like appear in our best theories; but the role they play in calculation is only ever one of approximation. No-one has ever calculated a *true* area of a circle using π , for we do not have a ‘true’ numerical value for π . Since the only role played is one of approximation, all that the nominalist must do with the mathematical constants that are irrational, like π , is provide an approximation.

We have already seen how to express ratios; π will prove no exception. We know, of course, that the circumference of a circle is equal to π multiplied by the diameter of the circle. To give an idea of the way in which we will then make sense of π , simply rearrange the formula: $C/D = \pi$. So, suppose that we took the value of π to be 3.142. In that case, simply take a circle that has a circumference of 3142 points, and a diameter of 1000 points. Any multiplication by π would then simply require us to multiply by 3142 and divide by 1000 (in the manner in which multiplication by fractions requires). If we require a more precise value of π we need simply consider a larger circle; one that has, for instance, 314159265358979323846 points in its circumference, and a diameter of 1000000000000000 points. You want more precision? Consider a larger circle. This strategy can, I think, be used to handle any mathematical constant that is irrational. We can’t quite get at *it* (the mathematical constant in question); but we can get at a very close approximation. Since very close approximations are all that we calculate with, the nominalist can preserve those calculations.⁹

6 Explanation and ontology

In this section of the paper I want to consider the wider implications of the paraphrase, as well as the sense in which it helps us to provide an explanation of the life-cycle of the cicada. Suppose that we allow that the paraphrase is adequate. There remain two questions to which we should want answers. First, for everything that has been said so far, it remains unclear as to the extent of the ontological commitments of the proposal. For instance, Lewis (1993) is explicit in his paraphrase of set theory that a fully adequate paraphrase of all of the axioms of set theory may require us to posit a

‘(strongly) ‘inaccessible’ infinity of atoms—an infinity that transcends our commonplace alephs and beths in much the same way that any infinity transcends finitude. There will be inaccessibly many atoms, inaccessibly many singletons, and inaccessibly many sets’ (Lewis 1993, p. 23).

⁸ I leave the proof to the reader.

⁹ I concede that nothing that I’ve said here speaks to the explanatory role attributed to phase spaces by Lyon and Colyvan (2008).

It is all very well, then, for me to offer this paraphrase of prime number talk, but how does this connect up to the Lewisian paraphrase, and what are its attendant ontological commitments? One might reason as follows: ‘there are infinitely many prime numbers’ is true; as a consequence, I am committed to (at the very least) an infinity of objects. This might be bearable. If we’re admitting extensionless space-time points into our ontology then we’re already committed to there being an infinity of them within any extended region of space and time.¹⁰ But it would be nice if we were not *committed* to this result. In Sect 6.1 I explore one option that the nominalist has available to them if they endorse the paraphrase and I will assume, for the most part, that we wish to avoid a commitment to points (in footnotes at various points in what follows I explain how a point-metaphysic might make the solution more elegant).

Second, suppose that the paraphrase turns out to be formally adequate and the ontological commitments bearable: nonetheless, it might still be objected that we have yet to show that the paraphrase is *fit for purpose*. The task in hand is not one that requires of us mere formalism. What we require is an *explanation* of the fact that primes minimize intersection. So, where is the explanation? If I cannot point to the explanation, then the paraphrase is of no value in response to the enhanced indispensability argument. I turn to this issue in Sects. 6.2, 6.3, 6.4, and 6.5, though I rely upon my chosen account of explanation in dealing with the question of ontological commitment in Sect. 6.1.

6.1 Ontology

Let us begin with the ontological commitments of the paraphrase. In response to the second premise of Baker’s argument we are supplying a nominalist explanation of why the periodic cicada has evolved to have a life-cycle that is prime. This specific challenge need not require of us that we give a complete paraphrase of *all* mathematical claims; all that the argument requires of us is that we give a nominalistic explanation of the life-cycle of the cicada. Because that life-cycle is prime, and it seems that primeness plays an indispensable explanatory role, so we must give a paraphrase of what it is to be prime that then explains why said lifecycle is prime. We also saw that it is the various number-theoretic lemmas that look to explain the reason for primes minimizing intersection, and so the enhanced indispensability argument forces us to offer a paraphrase of these, too.

But there is no attendant requirement to then paraphrase all of the putative mathematical truths. After all, the claim made in the first premise of the enhanced indispensability argument is that we ought rationally to believe in the existence of any entity that plays an indispensable role in our best scientific theories. This does not require of us that we posit any entities over and above those that play such an explanatory role.

Somewhat cautiously, I do not think that ‘there are infinitely many prime numbers’ plays any *explanatory* role in our best science. (If the Platonist can show otherwise, then I will, obviously, be required to say more and thus the move to Lewis’ solution might be forced upon me.) Because of this I do not consider the sentence to be one in need of a paraphrase. This is, of course, precisely the same line that was taken

¹⁰ For discussion, see, e.g. Arntzenius (2008).

in response to talk of π , above. We can paraphrase *to the extent that is required for calculation and explanation in our best science*: I make no further promises. And of course this enables me to radically reduce our ontological commitments.

Indeed, I agree with Saatsi when he notes that the function of mathematical discourse is to *express* facts about the physical world. If we are supposing that the physical world does not include an infinity of concrete objects then the fact expressed by the paraphrase is, itself, not a fact about an infinity of objects and so the paraphrase *should not* treat it the claim that ‘there are infinitely many prime numbers’ as one that is true. To make sense of this we must now turn our attention to the role the second concern, that of explanation.

6.2 Explanation

Thus, the second concern; the lack of any obvious *explanation* in the paraphrase. I think that the right place to locate the explanation takes its cue from Saatsi. Ultimately, however, although I agree with the general shape of Saatsi’s solution, I think that more should be said.

Saatsi begins by asking why honeycombs are always hexagonal. The explanation, borrowed from Lyon and Colyvan, is this:

[T]he biological part of the explanation is that those bees which minimise the amount of wax they use to build their combs tend to be selected over bees that waste energy by building combs with excessive amounts of wax. The mathematical part of the explanation then comes from what is known as the honeycomb conjecture: a hexagonal grid represents the best way to divide a surface into regions of equal area with the least total perimeter. (Lyon and Colyvan 2008, p. 230)

Saatsi disputes that this is genuinely an instance of *mathematical* explanation. I quote him at length:

My preferred analysis of mathematics’ role here goes as follows. Instead of speaking of the ‘mathematical part’ of the explanation we can say that (a) mathematics can be used to represent all physically possible spatial configurations of honeycomb walls, and (b) by virtue of this representational capacity we can infer that the area of honeycomb walls is minimised with hexagonal cells. What we infer is a physical fact: in Euclidean physical space the hexagonal construction yields the minimum wall area. Mathematics may be indispensable for getting to know this crucial physical fact to which the full evolutionary explanation of the phenomenon then appeals. There is no “mathematical part” to this evolutionary explanation. Rather, mathematics only plays a role in representing physical facts concerning areas and volumes in Euclidean space, allowing us to infer certain physical facts from other physical facts, and hence providing us knowledge of the crucial explanatory physical fact (Saatsi 2011, pp. 145–146)

According to Saatsi, what the mathematics does is describe (or represent) a physical fact about space: in Euclidean physical space hexagonal structure minimises the wall

area. But that's all that there is to it. And this is a physical fact. Saatsi thinks that matters are very similar in the cicada case. Here, all that is explained by the mathematics is a fact about time; that particular collections of years overlap in ways that others do not. Because the mathematical formalism is used to represent physical facts, so it is easy to see that physical facts are doing all of the explanatory work here.

I think this is correct. But, if it is, then why should we bother with this rather long and involved paraphrase? We must now turn to Saatsi's discussion of the cicada case.

Saatsi's suggestion (2011, p. 149) in the cicada case is that we replace premises (5) and (6) in Baker's argument (above) with the following:

(5/6)* For periods in the range 14–18 years the intersection minimizing period is 17 [fact about time]

Saatsi (2011, p. 150) then considers the following objection to the supplementation of (5) and (6) with (5/6)*. Unlike (5/6)*, Baker's original explanation is,

a good explanation because it unifies these two phenomena under a single 'argument pattern', and (relatedly) it can be generalized to other actual or hypothetical cases. For example, it predicts that other organisms with periodical life cycles are also likely to have prime periods. It is therefore better than any historico-ecological explanation that concatenates two separate and independent explanations of the two different period lengths. Hence by inference to the best explanation, we ought to believe in the entities invoked in the number theoretic explanation, which includes abstract mathematical objects such as numbers. (Baker 2009, p. 621)

By way of response, Saatsi (2011, p. 152) thinks that we can offer up a fully general 'argument pattern'.

- (4) having a life-cycle which minimizes intersection with other (nearby/lower) periods is evolutionarily advantageous. (biological law)
- (5/6)** There is a unique intersection minimizing period T_x for periods in the range $[T_1, \dots, T_2]$ years [fact (?) about time]
- (7) Cicadas in the ecosystem-type, E, are limited by biological constraints to periods from T_1 to T_2 years. [ecological constraint]
- ...
- (8) Cicadas in ecosystem-type, E, are likely to evolve T_x -year periods.

This physical fact about time—mentioned in (5/6)** is what Saatsi thinks is represented by the mathematical formalism. This new premise (5/6)**, Saatsi thinks, has as much explanatory power as the original and, crucially, does not make reference to numbers. I do not think this fully adequate.

In Sect. 6.3 I argue that there are predictions that can be made using the paraphrase that cannot be made using Saatsi's model and that this greater predictive power is a point in favour of the paraphrase; in Sect. 6.4 I argue that this suggests a way in which the paraphrase has greater explanatory power than does Saatsi's model and, in Sect. 6.5, I defend the claim that greater predictive and explanatory power is a genuine advantage of the paraphrase.

6.3 Prediction

It is Saatsi's contention that the above argument pattern permits us, just as Baker's does, to make predictions. We can, for example, make predictions about what will happen when we have a range of cases open to us with life-cycles from 20 to 24 years—we will find that there is a particular life-cycle that minimises intersection.

This point is important. Suppose we allow Saatsi that it is the physical facts that are expressed by mathematical claims. If we then showed that there were correct predictions that could be made using Baker's model that could not be made using Saatsi's, then this would point to the superiority of Baker's model. Likewise, if we can make a prediction using the paraphrase, that cannot be made with Saatsi's account, then that would point to the superiority of my account.

There is, crucially, a prediction that we could make using materials from my case, that we cannot with Saatsi's. Suppose we consider Saatsi's stick-case (2011, p. 150). The number theoretic claim deployed in Baker's argument, that primes minimize intersection, could be used to predict that *sticks* of equal length, laid end to end, will tend to minimize intersection (if they have a range of lengths from 14 to 17 cm) iff the length of the sticks is such as to be prime. Saatsi could not predict that using $(5/6)^{**}$; $(5/6)^{**}$ is explicitly *a claim about years* and the ways in which years minimise intersection. Facts about years do nothing to explain why sticks of various spatial lengths should minimise intersection.

With this in hand we ought to turn our attention to three matters. First, notice that the paraphrase I've offered does generate the right prediction, for we can just as easily treat the F's in question as sticks as we can years and so we can use this to predict that sticks of equal length, laid end to end, will tend to minimize intersection if they have a particular length in the range of lengths from 14 to 17 cm. Thus it appears that the proposed paraphrase fares just as well as Baker's Platonist account when it comes to consideration of predictions about time, space and the like.

The second point that is in need of consideration is whether or not Baker's model, in turn, permits us to make more true predictions than does the paraphrase. One might take the view that it does. For instance, suppose that there are n particulars in existence. We might still make the prediction that there will be a unique number of particulars, between n and $n + m$, that minimises intersection. This is something that can be predicted by Baker's model, for the number theoretic lemmas do not place an upper bound on the prime numbers. In contrast, the paraphrase, as already discussed, does appear to place an upper bound on the mathematical truths'; it is limited by the extent of reality such that there are no mathematical truths about values greater than n .¹¹ This appears to leave Baker's model preferable.

Notice, however, that according to the nominalist Baker's model in fact makes a *false* prediction. Recall that according to the view sketched above not all mathematical claims *are true*. Since the claim under consideration is one that would only be true

¹¹ Of course, if we endorse the view that there are infinitely many space-time points then we will have a ready solution to this problem for there will be infinitely many points and so the upper bound will be infinity itself. Problems would only arise, then, when it comes to considering higher-order infinities; for instance, an uncountable infinity.

were there to exist more particulars than in fact exist, so it is a false prediction because as a matter of fact there are no such particulars.¹² As I said above (Sect. 6.1), I am not *committed* to recovering all of mathematics; only those parts of it that are indispensable to our best science and calculation and explanation therein. If the number of concrete objects is n , then (again, assuming eternalism) it is quite simply false to say that, for instance, ‘there will be $n + 1$ objects’. It is, thus, no cost of the paraphrase if it does not preserve all of the predictions that Baker’s view does, for it simply insists that these predictions are false and that there is no calculation essential to our best theory that requires matters to be otherwise.¹³ Perhaps most pertinently given the dialectic under consideration here, I take the view that there is nothing in need of *explanation* in our best science that requires us to posit more objects than there are as a matter of contingent fact.¹⁴

Third, notice that the paraphrase permits us to make another (true) prediction about what will happen in another case. I think the additional case important. The cases involving sticks and years might be cases in which *the same physical fact* is expressed: a physical fact about the structure of an overarching space-time. Let me turn to a different case in order to block this line of resistance.

Consider the following. Rather than sticks, suppose that we have collections of electrons. Imagine that we have various different groups of electrons, consisting of groups of 14 electrons, 15 electrons, 16 electrons, 17 electrons and 18 electrons. We also have equinumerous groups of positively charged protons. Our challenge is to generate a system with neutral charge using the various different collections of particles. The only conditions are these: (i) each group of electrons must be equinumerous; (ii) each group of protons must be equinumerous; (iii) we must use at least a group of electrons; (iv) we may not subdivide the groups. Given the paraphrase I offered, we can predict that we will require more particles in cases where we’re dealing with groups of 17 electrons than when we’re not—because the lcm of prime groups is higher than those of non-prime groups.

What we will find is that we tend to require more particles when we’re dealing with groups of 17 electrons than when we’re not. In other words, the lcm tends to be higher when we have 17 electrons. The paraphrase enables us to make a prediction where Saatsi’s model does not.

6.4 Explaining the isomorphism

A natural question arises in connection with the previous cases: ‘*why* do we find that prime numbers of objects minimise intersection in both cases?’ If, as per Saatsi’s claim, the function of mathematics is to describe the structure of reality, then what

¹² It *helps* here to assume eternalism about the philosophy of time such that all particulars past, present and future, exist.

¹³ Though, again, an infinity of space-time points presents the proponents of the paraphrase with an easier way out of this problem at the expense of endorsing the salient metaphysic.

¹⁴ At least, if there is an associated cost it is that we are required to think some mathematical claims false. There are ways of ameliorating that cost, however (e.g. Dorr 2008), and, or so it seems to me, it is better to have this cost than to endorse Platonism and endorse the attendant costs there, e.g. Liggins (2006).

physical structure of reality is it that ensures that sticks, years, and charge, all behave in the same way? Whence the isomorphism between the cases?

This question seems richly suggestive. It appears that there is an isomorphism between the temporal case, the spatial case and the case involving the particles. If the function of mathematics is to represent physical facts and the mathematics in the three cases is isomorphic, then we should naturally *expect* there to be a single physical fact being represented by the mathematics used in the three cases—else we should have some other explanation of why the mathematics in the two cases is isomorphic. Saatsi does not provide any explanation of this isomorphism and so it is an unexplained coincidence, according to his view, that prime numbers of *any* particular minimise their intersection. I assume that a theory that leaves such a phenomena as unexplained is, ultimately, inferior to one that provides an explanation. That being the case, the provision of the paraphrase is apt.

Indeed, it seems quite reasonable to *expect* there to be an explanation for this isomorphism. After all, both cases seem to have something in common: namely, that they require us to deal with prime numbers of entities. And, of course, an explanation for the isomorphism is readily available for the Platonist. The reason that collections of entities overlap less frequently when they are in collections that are prime is a fact about primeness; primeness explains the infrequent overlap of collections of entities, via the number-theoretic explanation posited in the lemmas.

The paraphrase that I have offered on behalf of the nominalist permits them to explain the isomorphism; what explains the infrequent overlap of collections of entities that are prime is a fact about primeness, where primeness is understood as being a particular property of mereological fusions. Thus, where the Platonist has a fact about number explain, via the lemmas, the infrequent overlaps of prime collections of entities, the paraphrase is engaged in the task of describing the physical facts that, in turn, explain why primes minimise intersection.

In order to specify what structure is described in the world, by calling the Fs ‘prime’, we must offer a nominalistically acceptable paraphrase of prime-number talk. This slow, dry and highly unglamorous route to nominalism is not easy; but it is the right road.

6.5 The virtues of explanatory and predictive power

Before I leave this line of thought, it’s worth tying off one loose end: why think that the (alleged) greater explanatory and predictive power of the paraphrase makes it preferable to the account that Saatsi has given? We might, for instance, be of the mind that the greater explanatory and predictive power is all well and good: but why think that a *genuine* advantage of the paraphrase?

An opponent taking such a line is extreme; we do typically think that explanatory and predictive power are taken to be theoretical virtues. Nonetheless, it would be good if we could say something about *why* a theory, T, that offers more explanatory and predictive power than another theory T*, is thereby preferable to T*. It is with such an aim in mind that I borrow a line of argument from Huemer (2009).

Huemer, in considering whether or not we should prefer simple theories to their more complex counterparts, argues in favour of simple theories. The method of justification that he prefers in this task is what he calls the ‘Likelihood account’. In Huemer’s (2009, p. 221) words:

The essential point is that typically a simple theory can accommodate fewer possible sets of observations than a complex theory can: the simple theory makes more specific predictions. The realization of its predictions is consequently more impressive than the realization of the relatively weak predictions of a complex theory.¹⁵

However, we are not engaged, here, upon an analysis of simplicity. It is therefore unclear as to how these remarks are salient.

Permit me to rid us of this lack of clarity. Just as Huemer claims that a simple theory can accommodate fewer possible sets of observations, it is clear that the paraphrase can accommodate fewer possible sets of observations than Saatsi’s revised premise in the cicada case. In the cicada case, Saatsi suggests that it is a fact about time that prime periods minimise intersection. That fact is, of course, perfectly consistent with prime numbers of particles *not* minimising intersection in the proton/electron case discussed above (Sect. 6.3). A fact about time does not commit us to a fact about the ways in which protons and electrons overlap one another, after all.

In contrast, the paraphrase that I’ve offered is unable to accommodate a putative observation of prime numbers of particles not minimising intersection. That the paraphrase cannot accommodate such an observation is obvious: the paraphrase treats the minimising of intersection as a fact of *mereology*. As a consequence, the paraphrase is compatible with fewer items of data than is Saatsi’s account and, assuming that Huemer’s reason for preferring simpler theories is sound, the paraphrase is preferable to Saatsi’s account.¹⁶

7 Conclusion

We can paraphrase talk of primes in a fashion that is acceptable to nominalists and we can offer a nominalistically acceptable paraphrase of the explanation of the cicada case that Baker offers us. To repeat, this is not necessarily to say that we should endorse the ensuing account of primes, but, *contra* Baker, there is certainly such an account available. Thus we see that Baker’s claim that there is a mathematical entity playing an indispensable role in our best sciences, has been undermined.

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¹⁵ Huemer goes on to offer a good deal of justification for this claim. I do not repeat this here but recommend it to the reader—especially pp. 221–224.

¹⁶ I don’t see that the paraphrase is similarly advantaged with respect to Baker’s explanation; both views make the same *number* of predictions, but give different truth-values to the predictions made (see Sect. 6.2) for discussion.

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