Ontario Math Circles
First Annual ARML Team Selection Test
Solutions

1. Determine the value of $\frac{6^2 \times 35^2}{14^2 \times 15^2}$.
   Answer. $\boxed{1}$

$$\frac{6^2 \times 35^2}{14^2 \times 15^2} = \frac{2^2 3^2 \times 5^2 7^2}{2^2 7^2 \times 3^2 5^2} = 1$$

2. Determine the number of ways to place three As, three Bs, and three Cs in a 3×3 grid such that each row and column contain one of each letter.
   Answer. $\boxed{12}$
   Without loss of generality, assume that the top left corner contains A. There are two ways to place the remaining As, two ways to place the Bs, and one way to place the Cs for a total of $(2)(2)(1) = 4$ ways. Taking into account the starting assumption, there are $(3)(4) = 12$ ways to fill the grid.

3. Determine all values of x that satisfy the equation

$$x^2 - 4034x + 4068289 = 0$$

   Answer. $\boxed{2017}$
   The given equation is a perfect square,

$$x^2 - 4034x + 4068289 = (x - 2017)^2 = 0$$

   Therefore, $x = 2017$ is the only solution to this equation.

4. Let A, B, and C be three non-collinear points and M be a point on AB. Let P and Q be two points on the plane such that PM is the angle bisector of $\angle CMA$ and QM is the angle bisector of $\angle CMB$. Determine the measure of $\angle PMQ$.
   Answer. $\boxed{90^\circ}$
   Since $\angle CMQ = \frac{1}{2}(180^\circ - 2\angle PMC) = 90^\circ - \angle PMC$. Thus,

$$\angle PMQ = \angle PMC + \angle CMQ = 90^\circ$$

5. Determine all real values of m such that

$$(m^2 - 1)x^2 + 1 = 2(m - 1)x$$

   has at least one real solution.
   Answer. $\boxed{(-\infty, 1)}$
   If $m = 1$ then $1 = 0$, which has no solutions. If $m = -1$ then $1 = -4x$, which has a real solution. If $m \neq \pm 1$ then using the discriminant,

$$(-2(m-1))^2 - 4(m^2 - 1)(1) = 8 - 8m \geq 0$$

   which is $m \leq 1$. Therefore, the range is $m < 1$.

6. An ant is traveling from the point $(0, 0)$ to $(4, 5)$. In each move, the ant can only move one unit to the right or one unit up. Determine the probability of the ant passing through $(2, 3)$.
   Answer. $\boxed{\frac{10}{21}}$
   There are a total of $\binom{9}{4}$ different paths from $(0, 0)$ to $(4, 5)$. There are $\binom{5}{2}\binom{4}{2}$ different paths from $(0, 0)$ to $(4, 5)$ which passes through $(2, 3)$. Therefore, the probability is

$$\frac{\binom{5}{2}\binom{4}{2}}{\binom{9}{4}} = \frac{(10)(6)}{126} = \frac{10}{21}$$

7. In a party, every person must shake hands with every other person exactly once. There are currently 10 people at the party, who have already shaken hands. 10 more people will arrive one after another, how many new handshakes will occur?
Answer. $\boxed{145}$
When the room has n people and a new person joins, there are n new handshakes. Thus, going from 10 people to 20 people will require

$$10 + 11 + \cdots + 19 = 10(10) + 1 + 2 + \cdots + 9 = 100 + \frac{9(10)}{2} = 145$$

8. 46 is a base 7 number, determine the equivalent base 3 number.
Answer. $\boxed{1021}$

$$46_7 = 4 \times 7 + 6 = 34 = 1 \times 3^3 + 2 \times 3 + 1 \times 3^0$$

Therefore, $46_7 = 1021_3$

9. Determine the number of ordered positive integral triplets (a, b, c) that satisfy the following system of equations

$$\begin{cases} abc = 2016 \\ (a-1)(b-1)(c-1) = 1573 \end{cases}$$

Answer. $\boxed{3}$
Without loss of generality, assume that $a \leq b \leq c$. Since $1573 = 11^2 \times 13$ then $a = 2$ or $a = 12$.
If $a = 2$ then

$$\begin{cases} bc = 1008 \\ (b-1)(c-1) = 1573 \end{cases}$$

This is not possible because

$$1008 = bc > (b-1)(c-1) = 1573$$

Thus, $a = 12$. In this case, $b = 12$ and $c = 14$. Finally, it remains to check that $12 \times 12 \times 14 = 2016$. Therefore, there are $\frac{3!}{2!} = 3$ possible solutions.

10. A rectangular prism has integral side lengths and has a volume of 9. Determine the sum of all possible surface areas of this rectangular prism.
Answer. $\boxed{68}$
Let the side lengths be x, y, z then $xyz = 9$. Observe that any permutation of x, y, z will yield the same surface area, thus assume that $x \leq y \leq z$.
If $x = 1$ then either $y = 1$ and $z = 9$ or $y = 3$ and $z = 3$. In the first case, the surface area is 38. In the second case, the surface area is 30.
If $x = 3$ then $yz = 3$, which is not possible because $x \leq y \leq z$.
Thus, there are two possible two surface areas, 38 and 30. Therefore, the answer is $30 + 38 = 68$.

11. Determine the sum of all real solutions to the equation $\sin x = x^{2017}$.
Answer. $\boxed{0}$
Let $f(x) = \sin x - x^{2017}$. Observe that $f(x) = -f(-x)$, which means that $f(x)$ is an odd function. Thus, every positive real solution to $f(x) = 0$ will be mirrored to a negative real solution. Therefore, the sum of all real solutions must be 0.

12. Let x_1 be a solution to the equation

$$2x + 2^x = 8$$

and let x_2 be a solution to the equation

$$2x + 2\log_2(x-1) = 4$$

Determine all possible values of $x_1 + x_2$.
Answer. $\boxed{4}$
In the first equation, by inspection, $x = 2$ is a solution. Observe that the left hand side is strictly increasing but the right hand side is a constant. Therefore, there is exactly one solution. A similar argument can show that $x = 2$ is the unique solution to the second equation. Therefore, $x_1 + x_2 = 4$.

13. Given that $\frac{\sin(x+y)}{\sin(x-y)} = 2017$, determine the value of $\frac{\tan x}{\tan y}$.
Answer. $\boxed{\dfrac{1009}{1008}}$
Observe that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x \cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y} = 2017$$

which simplifies to

$$2018 \cos x \sin y = 2016 \sin x \cos y$$

Therefore,

$$\frac{\tan x}{\tan y} = \frac{\sin x \cos y}{\cos x \sin y} = \frac{1009}{1008}$$

14. Let $\frac{a}{b+c} = \frac{b}{a+c} = \frac{c}{a+b}$ where a, b, c are real numbers. Determine the sum of all possible values of $\frac{a}{b+c}$.
Answer. $\boxed{-\dfrac{1}{2}}$
Let $x = \frac{a}{b+c}$. If $a+b+c \neq 0$ then the equations are equivalent to

$$\begin{cases} (b+c)x = a \\ (a+c)x = b \\ (a+b)x = c \end{cases}$$

Adding these three equations yields $2(a+b+c)x = a+b+c$. Therefore, $x = \frac{1}{2}$.
If $a+b+c = 0$ then $a+b \neq 0$, $b+c \neq 0$, and $a+c \neq 0$. If any of a, b, c is 0 then they all must be 0, which is a contradiction. Thus, none of a, b, c are zero. Therefore, $a+b = -c$, $b+c = -a$, and $a+c = -b$, which implies that $x = -1$. The final answer is $\frac{1}{2} - 1 = -\frac{1}{2}$.

15. Find the nearest integer to $(17 - 12\sqrt{2})^{-0.5}$.
Answer. $\boxed{6}$

$$\begin{aligned}(17 - 12\sqrt{2})^{-0.5} &= \frac{1}{\sqrt{\left(3 - 2\sqrt{2}\right)^2}} \\ &= \frac{1}{3 - 2\sqrt{2}} \\ &= 3 + 2\sqrt{2} \\ &\approx 3 + 2(1.5) \\ &= 3 + 3 \\ &= 6\end{aligned}$$

Thus, the nearest integer is 6.
Remark. The approximation for $\sqrt{2} \approx 1.5$ is very crude. Using 1.414 would be much better.

16. Let a, b, and c be positive integer numbers such that $c \geq a$ and $a + c = b$. Determine the largest possible value of the smaller root of $ax^2 + bx + c = 0$.
Answer. $\boxed{-1}$
Observe that

$$\begin{aligned}0 =& ax^2 + bx + c \\ 0 =& ax^2 + (a+c)x + c \\ 0 =& ax(x+1) + c(x+1) \\ 0 =& (x+1)(ax+c)\end{aligned}$$

One root is -1 and the other root is $-\frac{c}{a} \leq -1$. Therefore, the largest possible value of the smaller root is -1. This can be obtained with $x^2 + 2x + 1 = 0$.

17. Let P be the point $(3, 2)$. If A is a point on the line $y = x$ and B is a point on the $x-$axis such that the perimeter of $\triangle ABP$ is minimized, determine the midpoint of the line segment AB.

Answer. $\boxed{\left(\frac{143}{60}, \frac{13}{12}\right)}$

Reflecting P in the line $y = x$ yields $P_1 = (2, 3)$. Reflecting P in the x-axis yields $P_2 = (3, -2)$. Observe that the minimum perimeter of $\triangle ABP$ is the same as the length of P_1P_2. The equation of P_1P_2 is

$$y = -5x + 13$$

This intersects the $x-$axis at $\left(\frac{13}{5}, 0\right)$ and intersects the line $y = x$ at $\left(\frac{13}{6}, \frac{13}{6}\right)$. Therefore, the midpoint of A and B is

$$\frac{1}{2}\left(\left(\frac{13}{5}, 0\right) + \left(\frac{13}{6}, \frac{13}{6}\right)\right) = \left(\frac{143}{60}, \frac{13}{12}\right)$$

Remark. To make this solution more rigorous, it remains to show that this is the optimal path. This can be easily done by using the triangle inequality.

18. Let S be the set of all four digits numbers such that each digit is either $1, 2,$ or 3. Determine the number of ways to select three distinct numbers from S such that, for $i = 1, 2, 3, 4$, the i^{th} digit in all three numbers are either all equal or all unique. For example,

$$1111, 2221, 3331$$

is one way to select the three numbers.

Answer. $\boxed{1080}$

Observe that once two numbers are selected from S, the third number is uniquely determined. Taking into over counting, there are

$$\frac{1}{3}\binom{3^4}{2} = \frac{1}{3}\binom{81}{2} = 1080$$

Remark. This problem was proposed by Lillian Zhang.

19. Let $x > 1$ and

$$(\log_x 128)(\log_{128} 16) = \sqrt[3]{72}$$

determine the value of $\log_{\frac{2}{x}} 256$.

Answer. $\boxed{72 + 48\sqrt[3]{3} + 32\sqrt[3]{9}}$

The given equation is equivalent to

$$\log_x 16 = 2\sqrt[3]{9}$$

Thus,

$$\begin{aligned}
\log_{\frac{2}{x}} 256 &= \frac{\log_x 16^2}{\log_x \frac{2}{x}} \\
&= \frac{2\log_x 16}{\log_x 2 - \log_x x} \\
&= \frac{4\sqrt[3]{9}}{\frac{1}{4}\log_x 16 - 1} \\
&= \frac{16\sqrt[3]{9}}{2\sqrt[3]{9} - 4} \\
&= \frac{8\sqrt[3]{9}}{\sqrt[3]{9} - 2} \\
&= \frac{8\sqrt[3]{9}\left(\sqrt[3]{9^2} + 2\sqrt[3]{9} + 4\right)}{\left(\sqrt[3]{9} - 2\right)\left(\sqrt[3]{9^2} + 2\sqrt[3]{9} + 4\right)} \\
&= \frac{72 + 48\sqrt[3]{3} + 32\sqrt[3]{9}}{9 - 8} \\
&= 72 + 48\sqrt[3]{3} + 32\sqrt[3]{9}
\end{aligned}$$

20. The current time on a clock face is 7AM. Determine the next time, rounded to the nearest second, such that the minute hand and the hour hand are on top of each other. Express your answer in the format of Hours:Minutes:Seconds.
Answer. $\boxed{7:38:11}$
Every second, the hour hand moves $\frac{30}{3600} = \frac{1}{120}$ degrees. Thus, after t seconds, the clockwise angle from the 12 to the hour hand is

$$210 + \frac{1}{120}t$$

degree.
Every second, the minute hand moves $\frac{360}{3600} = \frac{1}{10}$ degrees. Thus, after t seconds, the clockwise angle from the 12 to the minute hand is

$$\frac{1}{10}t$$

degree.
Equating these two values and solving for t yields

$$t = 2290.\overline{90} \approx 2291 = 38(60) + 11$$

Therefore, the desired time is $7:38:11$.

21. Determine the sum of the prime factors of 14541.
Answer. $\boxed{171}$
This number can be encoded into a polynomial as

$$\begin{aligned}
&x^4 + 4x^3 + 5x^2 + 4x + 1\\
=&x^4 + 4x^3 + 6x^2 + 4x + 1 - x^2\\
=&(x+1)^4 - x^2\\
=&\left((x+1)^2 - x\right)\left((x+1)^2 + x\right)\\
=&\left(x^2 + x + 1\right)\left(x^2 + 3x + 1\right)
\end{aligned}$$

Evaluating this at $x = 10$ yields $(111)(131) = (3)(37)(131) = 14541$. Therefore, $3 + 37 + 131 = 171$.

22. Let x and y be real numbers such that $2x + y = 1$ and $|y| \leq 1$. Determine the minimum possible value of $2x^2 + 16x + 3y^2$.
Answer. $\boxed{3}$
Since $2x + y = 1$ and $|y| \leq 1$ then $0 \leq x \leq 1$. Therefore,

$$\begin{aligned}
&2x^2 + 16x + 3y^2\\
=&2x^2 + 16x + 3(1-2x)^2\\
=&14x^2 + 4x + 3\\
=&14\left(x + \frac{1}{7}\right)^2 + \frac{19}{7}
\end{aligned}$$

On $0 \leq x \leq 1$, the minimum value is obtained at $x = 0$, which is 3.

23. The following two curves

$$\begin{cases} y = (x-1)^2 \\ x = (y-1)^2 \end{cases}$$

intersects at four distinct points. Determine the area of the quadrilateral formed by these four points.
Answer. $\boxed{\sqrt{5}}$
Observe that $(0,1)$ and $(1,0)$ are two solutions. Since the two quadratics are symmetric in the line $y = x$, they must have two intersections on $y = x$. Setting $y = x$ in the first equation yields the solution $\left(\frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$ and $\left(\frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right)$. Listing these four solutions in order,

$$(1,0), \left(\frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right), (0,1), \left(\frac{3-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right)$$

This figure is symmetric about the line $y = x$. By splitting the area accordingly, the area of the quadrilateral is $\sqrt{5}$.
Remark 1. Instead of making the observation that $(1, 0)$ and $(0, 1)$ are two solutions, an alternative would be to just use substitution and factor the resulting degree 4 polynomial.

24. Given that the following equation

$$\frac{1}{x^2 - 10x - 29} + \frac{1}{x^2 - 10x - 45} - \frac{2}{x^2 - 10x - 69} = 0$$

has exactly two solutions, α and β. Determine the value of $\alpha\beta$.
Answer. $\boxed{-39}$
Let $y = x^2 - 10x - 45$. The given equation becomes

$$\frac{1}{y + 16} + \frac{1}{y} = \frac{2}{y - 24}$$

Solving this yields $y = -6$. Therefore,

$$x^2 - 10x - 39 = 0$$

By Vieta's Formula, $\alpha\beta = -39$.

25. Let $ABCD$ be a rectangle with $\overline{AD} = 12$ and $\overline{AB} = 5$. Let P be the point on the line segment $\overline{AD}$ such that $\left|\overline{AP}\right| = \sqrt{2}$. Denote h_1 to be the distance from P to AC and h_2 to be the distance from P to BD. Determine the value of $h_1 + h_2$.
Answer. $\boxed{\frac{60}{13}}$
Let E be the point on $\overline{BD}$ such that $\overline{PE}$ is perpendicular to $\overline{BD}$. Similarly, let F be the point on $\overline{AC}$ such that $\overline{PF}$ is perpendicular to $\overline{AC}$. Let O be the intersection of the diagonals. Since $\left|\overline{DO}\right| = \left|\overline{AO}\right|$ then

$$\begin{aligned}(h_1 + h_2) \times \left|\overline{AO}\right| &= h_1 \times \left|\overline{DO}\right| + h_2 \times \left|\overline{AO}\right| \\ &= 2\left(|\triangle DPO| + |\triangle APO|\right) \\ &= \frac{1}{2}|ABCD| \\ &= 30\end{aligned}$$

By Pythagorean theorem, $\left|\overline{AO}\right| = \frac{13}{2}$ then $h_1 + h_2 = \frac{60}{13}$.
Remark 1. Observe that the length of $\overline{AP}$ was actually useless.
Remark 2. An alternative solution which requires no insight would be to use analytical geometry.

26. Determine the largest integer N such that $a^{13} - a$ is divisible by N for every integer $0 \leq a \leq N - 1$.
Answer. $\boxed{2730}$
Let p be a prime that divides N then

$$a^{13} \equiv a \mod p$$

By Fermat's Little Theorem, the possible values of p are $2, 3, 5, 7, 13$. Thus,

$$N = 2^{d_1}3^{d_2}5^{d_3}7^{d_4}13^{d_5}$$

If one of the d_i's is greater than 1 then

$$a^{13} \equiv a \mod p^2$$

where $p \in \{2, 3, 5, 7, 13\}$. However, this modular equation cannot be true because $a = p$ is a contradiction. Therefore, $d_1 = d_2 = d_3 = d_4 = d_5 = 1$. Thus, $N = (2)(3)(5)(7)(13) = 2730$.

27. Let a, b, c be complex numbers such that

$$|a| = |b| = |c| = 2a + 2b + 2c = abc = 1$$

Determine one possible triplet (a, b, c).

Answer. $\boxed{\left(1, -\frac{1}{4} + \frac{i\sqrt{15}}{4}, -\frac{1}{4} - \frac{i\sqrt{15}}{4}\right)}$

Taking the conjugate of $a + b + c = \frac{1}{2}$ yields

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$$

Combining this with $abc = 1$ yields $ab + bc + ca = \frac{1}{2}$. By Vieta's Formula or simply brute force solving the system of equations

$$\begin{cases} a + b + c = \frac{1}{2} \\ ab + bc + ca = \frac{1}{2} \\ abc = 1 \end{cases}$$

yields $x^3 - \frac{1}{2}x^2 + \frac{1}{2}x - 1 = 0$. This can be factored as

$$(x - 1)\left(x - \left(-\frac{1}{4} + \frac{i\sqrt{15}}{4}\right)\right)\left(x - \left(-\frac{1}{4} - \frac{i\sqrt{15}}{4}\right)\right) = 0$$

Therefore, all desired triplets are the permutations of $\left(1, -\frac{1}{4} + \frac{i\sqrt{15}}{4}, -\frac{1}{4} - \frac{i\sqrt{15}}{4}\right)$.

28. Let a be a solution to the equation $2x^2 - 3x - 1 = 0$ and let b be a solution to the equation $x^2 + 3x - 2 = 0$. If $ab \neq 1$, determine the value of $\frac{ab+2017a+1}{b}$.

Answer. $\boxed{-1007}$

The second equation can be rewritten as

$$2\left(\frac{1}{b}\right)^2 - 3\left(\frac{1}{b}\right) - 1 = 0$$

Thus, $\frac{1}{b}$ and a are both solutions to the equation $2x^2 - 3x - 1 = 0$. Since $ab \neq 1$ then $a \neq \frac{1}{b}$. By Vieta's Formula,

$$a + \frac{1}{b} = \frac{3}{2}$$
$$\frac{a}{b} = -\frac{1}{2}$$

Therefore,

$$\frac{ab + 2017a + 1}{b} = a + \frac{1}{b} + 2017\frac{a}{b} = \frac{3}{2} - \frac{2017}{2} = -1007$$

29. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where a_i are non-negative integers for $i = 0, 1, \ldots, n$. If $f(1) = 21$ and $f(25) = 78357$. Determine $f(5)$.

Answer. $\boxed{677}$

Working in modulo 25 then $7 \equiv a_0$. Since $\sum_{i=0}^{n} a_i = 21$ then $a_0 = 7$. Thus,

$$78350 = a_1(25) + a_2(25)^2 + \cdots + a_n(25)^n$$

Dividing both sides by 25 yields

$$3134 = a_1 + a_2(25) + \cdots + a_n(25)^{n-1}$$

Working in modulo 25 again yields $a_1 = 9$. Repeating this process will yields $f(x) = 5x^3 + 9x + 7$. Therefore, $f(5) = 677$.

30. Let $f(x) = x^2 + 32x + 240$. Determine the product of all real solutions to the equation

$$f(f(f(x))) = 6545$$

Answer. $\boxed{247}$
Completing the square for $f(x)$ yields

$$f(x) = (x+16)^2 - 16$$

Substituting this version of $f(x)$ into the nested $f(x)$ yields

$$f(f(f(x))) = (x+16)^8 - 16$$

Equating this to 6545 yields the degree 8 polynomial equation

$$(x+16)^8 = 6561$$

The real solutions to this equation are $\pm 3 - 16$. The product of these two numbers is 247.

31. Determine the number of terms in the expansion of $(x+y+z)^{101}$ where the coefficient is divisible by 101.
Answer. $\boxed{5250}$
Recall that

$$(x+y+z)^{101} = \sum_{\substack{i+j+k=101 \\ i,j,k \geq 0}} \binom{2017}{i,j,k} x^i y^j z^k$$

where $\binom{101}{i,j,k} = \frac{101!}{i!j!k!}$. Since 101 is a prime number then the only coefficients that are not divisible by 101 are $\binom{101}{101,0,0}$, $\binom{101}{0,101,0}$, and $\binom{101}{0,0,101}$. By stars and bars, there are

$$\binom{101+2}{2} = (103)(51) = 5253$$

terms in the expansion of $(x+y+z)^{101}$. Therefore, there are $5253 - 3 = 5250$ terms that are divisible by 101.
Remark. An alternative solution is to consider the Pascal's Pyramid.

32. Let $X = (1+i)^{2016}$ be a complex number. Determine the number of digits of

$$\left\lfloor \frac{1}{2} \left| (X(1-i)) + \bar{X}(1+i) \right| \right\rfloor$$

Answer. $\boxed{304}$
Define A and B as

$$A = \frac{1}{2}\left(X + \bar{X}\right)$$
$$B = \frac{1}{2i}\left(X - \bar{X}\right)$$

Observe that $\frac{1}{2}\left((X(1-i)) + \bar{X}(1+i)\right) = A - B$ and that A is just the real part of X and B is the complex part of X. Observe that

$$\begin{aligned} X &= (1+i)^{2016} \\ &= \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{2016} \\ &= 2^{1008} \end{aligned}$$

Therefore, $A = 2^{1008}$ and $B = 0$. Thus, $\lfloor |A+B| \rfloor = 2^{1008}$. This contains

$$\begin{aligned}\left\lfloor \log_{10} 2^{1008} \right\rfloor + 1 &= \lfloor 1008 \log_{10} 2 \rfloor + 1 \\ &\approx \lfloor 1008(0.30103) \rfloor + 1 \\ &\approx \lfloor 303.44 \rfloor + 1 \\ &= 303 + 1 \\ &= 304\end{aligned}$$

digits

33. Let a, b, c be the lengths of the three sides of $\triangle ABC$ such that $a > b > c$, $2b > a + c$, and b is a positive integer. If $a^2 + b^2 + c^2 = 84$, determine the value of b.
Answer. $\boxed{5 \text{ or } 6}$
Since $a > b$ then

$$84 = a^2 + b^2 + c^2 > 2b^2$$

Thus, $b \leq 6$. Since b is an integer, its possible values are $1, 2, 3, 4, 5, 6$. For a, b, c to form a triangle, they must satisfy the following inequalities,

$$\begin{cases} a + c < 2b \\ c < b < a \\ a^2 + b^2 + c^2 = 84 \\ b + c > a \end{cases}$$

First and last inequality implies that $a < \frac{3}{2}b$. Next,

$$b^2 > c^2 = 84 - a^2 - b^2 > 84 - \frac{3}{2}b^2 - b^2 = 84 - \frac{5}{2}b^2$$

Thus, $b = 5$ or $b = 6$.
If $b = 5$ then

$$\begin{cases} a + c < 10 \\ c < 5 < a \\ a^2 + c^2 = 59 \\ 5 + c > a \end{cases}$$

Since $a + c = 10$ and $5 + c = a$ intersect at $(7.5, 2.5)$, which is outside of the circle such that the region

$$\{(a, c) : a + c < 10, 5 + c > a, c < 5 < a\}$$

contains a piece of the circle $a^2 + c^2 = 59$, then there must exist $a$and c that satisfy this problem. Although not required by the problem, it is possible to explicitly determine the range of possible values for a and c by solving the above system of inequality which is not that hard.
A similar analysis can show that $b = 6$ is also possible.

34. Determine the sum of all integers k such that $0 \leq k \leq 72$ and

$$\left\lfloor \sqrt{\frac{1}{k!} \prod_{n=1}^{72} n!} \right\rfloor - \sqrt{\frac{1}{k!} \prod_{n=1}^{72} n!} = 0$$

Answer. $\boxed{71}$
Observe that the equation is essentially saying that

$$\frac{1}{k!} \prod_{n=1}^{72} n!$$

is a perfect square. Using the identity $(2m)! = (2m-1)!(2m)$,

$$\begin{aligned}\prod_{n=1}^{72} n! &= (1! \times 3! \times 5! \times \cdots \times 71!)^2 \times 2 \times 4 \times \cdots \times 72 \\ &= (1! \times 3! \times 5! \times \cdots \times 71!)^2 \times 2^{64} \times 36! \\ &= (1! \times 3! \times 5! \times \cdots \times 71!)^2 \times 2^{64} \times 35! \times 6^2\end{aligned}$$

If $k \geq 37$, then the power of 37 in $\frac{1}{k!}\prod_{n=1}^{72} n!$ is an odd number, which is not possible.
If $k \leq 34$, then $\frac{35!}{k!}$ must be a perfect square. Observe that 35! has 5 multiples of 7. Therefore, for $\frac{35!}{k!}$ to be a perfect square, $k \leq 27$. This is impossible because $\frac{35!}{k!}$ would be divisible by 31 but not 31^2.
Therefore, $k = 35$ or 36, which implies that the final answer is $35 + 36 = 71$.

35. Determine the number of ordered integral triplets (a, b, c) with $0 \leq a, b, c \leq 100$ such that

$$a^3(b-c) + b^3(c-a) + c^3(a-b) = 0$$

Answer. $\boxed{30401}$
Define $f(a) = a^3(b-c) + b^3(c-a) + c^3(a-b)$. Since $f(b) = 0$ then $(a-b)$ is a factor of the original expression. Similarly, $(b-c)$ and $(c-a)$ are both factors of the original expression. Since the original expression is symmetric then

$$a^3(b-c) + b^3(c-a) + c^3(a-b) = A(a+b+c)(a-b)(b-c)(c-a) = 0$$

where A is a constant real number. It is obvious that $A \neq 0$. Thus, the problem is reduced to solving the equation

$$(a+b+c)(a-b)(b-c)(c-a) = 0$$

Since $a+b+c = 0$ is only possible when $a = b = c = 0$ then this piece can be discarded. Therefore, consider the equation

$$(a-b)(b-c)(c-a) = 0$$

If $a = b = c$ then this equation is true. There are 101 possible triplets for this. If exactly two of a, b, c are equal then this equation is true. There are

$$2\binom{101}{2}\binom{3}{2} = 30300$$

solutions. If a, b, c are all different then there is no solution. Therefore, there are $101 + 30300 = 30401$ possible triplets.

36. Let n be a positive integer. Consider the family of 46 curves

$$ky = x^2 + k^2$$

where $k = 45, 44, 43, \ldots, 2, 1, 0$. Determine the number of regions into which these 46 curves divide the plane.
Answer. $\boxed{2072}$
Let k_1, k_2, k_3 be three different number in the set $\{1, 2, \ldots, n\}$. Solving the first two equations yields $x = \sqrt{k_1 k_2}$ and solving the second two equations yields $x = \sqrt{k_2 k_3}$. If these two are the same then $k_1 = k_3$, which is a contradiction. Therefore, in the given family of curves, no three intersect at the same point.
By symmetry, consider the region when $x > 0$. This is equivalent to finding the number of regions 45 distinct lines can dissect a plane, which is known to be

$$\frac{n(n+1)}{2} + 1 = \frac{45(45+1)}{2} + 1 = 1036$$

Therefore, the entire plane is divided into $1036 \times 2 = 2072$.

37. Determine the number of permutations of $x_1, x_2, \ldots, x_{10}$ of the integers $-3, -2, -1, \ldots, 6$ that satisfy the chain of inequality

$$x_1x_2 \leq x_2x_3 \leq \cdots \leq x_9x_{10}$$

Answer. $\boxed{240}$

Solution. First note that no two negative numbers can occur consecutively because otherwise, either the resulting positive product would have to be followed by a non-positive product once the negatives are exhausted, or it would have to occur at the end but there are to many terms for that to be possible. Thus, the sequence must start with alternating positive and negative values. Each inequality $x_ix_{i+1} \leq x_{i+1}x_{i+2}$ implies that $x_i < x_{i+2}$ if $x_{i+1} > 0$ and $x_i > x_{i+2}$ if $x_{i+1} < 0$. The two main cases to consider are $x_1 = -3$ and $x_2 = -3$.

If $x_1 = -3$ then $x_3 = -2$, $x_5 = -1$ and either $x_6 = 0$ or $x_7 = 0$. If $x_6 = 0$ then the positive entries x_2, x_4, x_7, x_8, x_9 and x_{10} must satisfy $x_2 > x_4$, $x_9 > x_7$, and $x_8 < x_{10}$. Therefore are $\frac{6!}{2!^3} = 90$ such sequences. If $x_7 = 0$ then the positive entries x_2, x_4, x_6, x_7, x_8, and x_{10} must satisfy $x_2 > x_4 > x_6$ and $x_8 < x_{10}$ for $\frac{6!}{3!2!1!} = 60$ more sequences.

If $x_2 = -3$ then $x_4 = -2$, $x_6 = -1$, and either $x_7 = 0$ or $x_8 = 0$. If $x_7 = 0$ then the positive entries x_1, x_3, x_4, x_8, x_9, and x_{10} must satisfy $x_1 > x_3 > x_5$ and $x_8 < x_{10}$ for another 60 more sequences. If $x_8 = 0$, then the positive entries x_1, x_3, x_5, x_7, x_9, and x_{10} must satisfy $x_1 > x_3 > x_5 > x_7$ for $\frac{6!}{4!1!1!} = 30$ more sequences.

Therefore, there are $90 + 60 + 60 + 30 = 240$ possible sequences.

Remark. This is the exact same question as ARML 2016's individual #9.

38. Let ABC be an isosceles triangle with $\angle ABC = \angle ACB$. Let P be a point inside $\triangle ABC$ such that $\angle BCP = 30^\circ$, $\angle APB = 150^\circ$, and $\angle CAP = 39^\circ$. Determine the value of $\angle BAP$.

Answer. $\boxed{13^\circ}$

Start by constructing point Q such that $\triangle PQB$ is equilateral and Q is on the same side of $\overline{BP}$ as C. Observe that Q is the circumcenter of $\triangle BPC$. Since $\triangle ABC$ is isosceles then $\overline{AQ}$ is the perpendicular bisector of $\overline{BC}$. Since $\angle APB = 150^\circ$ then $\overline{AP}$ must be perpendicular to $\overline{BQ}$. Thus, $\overline{AP}$ must bisect $\angle BAQ$. Therefore, $\angle BAP = \frac{1}{3}\angle CAP = 13^\circ$.

39. Determine the smallest positive integer n such that for any set S of n real numbers, there exists a and b in S such that

$$0 < \frac{a-b}{1+ab} \leq 1$$

Answer. $\boxed{4}$

Let $S = \{x_1, x_2, x_3, x_4\}$ with $x_1 < x_2 < x_3 < x_4$. Let $\theta_i = \arctan(x_i)$ for each i so we have

$$-\frac{\pi}{2} < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \frac{\pi}{2}$$

Let $\phi_i = \theta_{i+1} - \theta_i$ for $i = 1, 2, 3$ and $\phi_4 = \pi + \theta_1 - \theta_4$. Since $\phi_i > 0$ for all i then

$$\sum_{i=1}^{4} \phi_i = \pi$$

Thus, $\phi_k \leq \frac{\pi}{4}$ for some k, which implies that $0 < \tan\phi_k \leq 1$. If $k \in \{1, 2, 3\}$ then

$$\tan\phi_k = \tan(\theta_{k+1} - \theta_k) = \frac{\tan\theta_{k+1} - \tan\theta_k}{1 + \tan\theta_{k+1}\tan\theta_k} = \frac{x_{k+1} - x_k}{1 + x_kx_{k+1}}$$

and if $k = 4$ then

$$\tan\phi_k = \tan(\pi + \theta_1 - \theta_4) = \frac{\tan\theta_1 - \tan\theta_4}{1 + \tan\theta_1\tan\theta_4} = \frac{x_1 - x_4}{1 + x_1x_4}$$

Observe that $n = 3$ is not possible by considering the set $\{-1.1, 0, 1.1\}$.

40. Determine all positive real numbers x such that

$$\lfloor 2x \rfloor, \lfloor x^2 \rfloor, \lfloor x^4 \rfloor$$

are the three side lengths of a non-degenerate triangle.

Answer. $\boxed{\left[\sqrt[4]{2}, \sqrt[4]{3}\right)}$

Observe that if $a \le b$ then $\lfloor a \rfloor \le \lfloor b \rfloor$.
If $x \in [0,1)$ then one of the side lengths is 0.
If $x \in \left[1, \sqrt[3]{2}\right)$ then $x^2 \le x^4 \le 2x$. If $x \in \left[1, \sqrt[4]{2}\right)$ then

$$\left\lfloor x^2 \right\rfloor + \left\lfloor x^4 \right\rfloor = 2 = \lfloor 2x \rfloor$$

which is a contradiction to the triangle inequality. If $x \in \left[\sqrt[4]{2}, \sqrt[3]{2}\right)$ then

$$\left\lfloor x^2 \right\rfloor + \left\lfloor x^4 \right\rfloor = 1 + 2 = 3 > 2 = \lfloor 2x \rfloor$$

This is a possible triangle.
If $x \in \left[\sqrt[3]{2}, 2\right)$ then $x^2 \le 2x \le x^4$. If $x \in \left[\sqrt[3]{2}, \sqrt[4]{3}\right)$ then

$$\left\lfloor x^2 \right\rfloor + \lfloor 2x \rfloor = 1 + 2 = 3 > 2 = \left\lfloor x^4 \right\rfloor$$

This is a possible triangle. If $x \in \left[\sqrt[4]{3}, \sqrt[4]{4}\right)$ then

$$\left\lfloor x^2 \right\rfloor + \lfloor 2x \rfloor = 1 + 2 = 3 = \left\lfloor x^4 \right\rfloor$$

which is a contradiction to the triangle inequality. If $x \in \left[\sqrt[4]{4}, \sqrt[4]{5}\right)$ then

$$\left\lfloor x^2 \right\rfloor + \lfloor 2x \rfloor = 2 + 2 = 4 = \left\lfloor x^4 \right\rfloor$$

which is a contradiction to the triangle inequality. If $x \in \left[\sqrt[4]{5}, 1.5\right)$ then

$$\left\lfloor x^2 \right\rfloor + \lfloor 2x \rfloor = 2 + 2 < 5 = \left\lfloor x^4 \right\rfloor$$

which is a contradiction to the triangle inequality. If $x \in \left[1.5, \sqrt[4]{6}\right)$ then

$$\left\lfloor x^2 \right\rfloor + \lfloor 2x \rfloor = 2 + 3 = 5 = \left\lfloor x^4 \right\rfloor$$

which is a contradiction to the triangle inequality. If $x \in \left[\sqrt[4]{6}, 2\right)$ then

$$\left\lfloor x^2 \right\rfloor + \lfloor 2x \rfloor \le 3 + 3 = 6 \le \left\lfloor x^4 \right\rfloor$$

which is a contradiction to the triangle inequality.
If $x \in [2, \infty)$ then $2x \le x^2 \le x^4$. Observe that

$$\lfloor 2x \rfloor + \left\lfloor x^2 \right\rfloor \le \left\lfloor 2x + x^2 \right\rfloor < \left\lfloor x^2 + 2x + 1 \right\rfloor = \left\lfloor (x+1)^2 \right\rfloor < \left\lfloor x^4 \right\rfloor$$

which is a contradiction to the triangle inequality.
Combining all possible ranges where the three lengths form a non-degenerate triangle yields $\left[\sqrt[4]{2}, \sqrt[4]{3}\right)$.
Remark. A rough solution to reach the answer much faster is to observe that after the interval $\left[\sqrt[4]{3}, \sqrt[4]{4}\right)$, if either $\left\lfloor x^2 \right\rfloor$ or $\lfloor 2x \rfloor$ is to increase by 1 then $\left\lfloor x^4 \right\rfloor$ will have to increase by at least 1. Therefore, no more triangles will be formed.